The Posterior metric and the Goodness of Gibbsianness for transforms of Gibbs measures

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Abstract

We present a general method to derive continuity estimates for conditional probabilities of general (possibly continuous) spin models subjected to local transformations. Such systems arise in the study of a stochastic time-evolution of Gibbs measures or as noisy observations.

We exhibit the minimal necessary structure for such double-layer systems. Assuming no a priori metric on the local state spaces, we define the posterior metric on the local image space. We show that it allows in a natural way to divide the local part of the continuity estimates from the spatial part (which is treated by Dobrushin uniqueness here). We show in the concrete example of the time evolution of rotators on the $(q-1)$-dimensional sphere how this method can be used to obtain estimates in terms of the familiar Euclidean metric.

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1 Introduction

The absence or presence of phase transitions lies at the heart of mathematical statistical mechanics of equilibrium systems. A phase transition in an order parameter that can be directly observed is of an obvious interest for the system under investigation. Moreover sometimes also the presence or absence of phase transitions is linked in a more subtle way to the properties of the system under investigation. In fact, it is understood that "hidden phase transitions" in an internal system that is not directly observable are responsible for the failure of the Gibbs property for a variety of important measures that appear as transforms of different sorts of Gibbs measures. For the mechanisms of how to become non-Gibbs and background on renormalization group type of pathologies and beyond, see the reviews [11, 8, 5].

Now, the first part of the analysis of an interacting system begins with an understanding of the "weak coupling regime" and proving results based on absence of phase transitions when the system variables behave as a perturbation of independent ones. There is a variety of competing ways to our disposition to do so, giving related but usually not equivalent results, notably Dobrushin's uniqueness theory [14, 1], expansion methods, and percolation and coupling methods.

Indeed, when it works, Dobrushin uniqueness has a lot of advantages, being not very technical, but very general, requiring little explicit knowledge of the system and providing explicit estimates on decay of correlations. Moreover, it implies useful properties generalizing those of independent variables. As an example of such a useful property we mention Gaussian concentration estimates of functions of the system variables which are obtained as a corollary when there is an estimate on the Dobrushin interaction matrix available [6, 7]. Especially when we are talking about continuous spin systems a Dobrushin uniqueness approach seems favorable, since cluster expansions are often applicable only with some technical effort [19, 12], and percolation and coupling are not directly available.

A particular interest has been in recent times in the study of the loss and recovery of the Gibbs property of an initial Gibbs measure under a stochastic time-evolution. The study started in [4] where the authors focussed on the evolution of a Gibbs measure of an Ising model under high-temperature spin-flip Glauber dynamics. The main phenomenon observed therein was the loss of the Gibbs property after a certain transition time when the system was started at an initial low temperature state. The measure stays non-Gibbs forever when the initial external field was zero. More complicated transition phenomena between Gibbs and non-Gibbs are possible at intermediate times when there is no spin-flip symmetry: The Gibbs property is recovered again at large but finite values of time in the presence of non-vanishing external magnetic fields in the external measure. A complete analysis of the corresponding Ising mean-field system in zero magnetic field was given in [3] where the authors analyzed the time-temperature dynamic phase diagram describing the Gibbs non-Gibbs transitions. In the analysis also the phenomenon of symmetry breaking in the set of bad configurations was detected, meaning that a bad configuration whose spatial average does not preserve the spin flip symmetry of the model appears.

What remains of these phenomena for continuous spins? The case of site-wise independent diffusions of continuous spins on the lattice starting from the Gibbs-measure of a special double-well potential was considered in [10]. It was shown therein that a similar loss of Gibbsianness will occur if the initial double-well potential is deep enough. In

contrast to the Ising model, this loss however is a loss without recovery, so the measure stays non-Gibbs for all sufficiently large times. This is due to the unbounded nature of the spins. Short-time Gibbsianness is proved to hold also in this model. While these results hold for a continuous spin model, the method of proof is nevertheless based on the investigation of a "hidden discrete model", exploiting the particular form of the Gibbs-potential. In [17] the authors studied models for compact spins, namely the planar rotor models on the circle subjected to diffusive time-evolution. It is shown therein that starting with an initial low-temperature Gibbs measure, the time-evolved measure obtained for infinite- or high-temperature dynamics stays Gibbs for short times and for the corresponding initial infinite- or high- temperature Gibbs measure under infinite- or high-temperature dynamics, the time-evolved measure stays Gibbs forever. Their analysis uses the machinery of cluster expansions, as earlier developed in [22]. Even before it was shown that the whole process of space-time histories can be viewed as a Gibbs measure[21]. This is interesting in itself, but does not imply that fixed-time projections are Gibbs.

Short-time Gibbsianness in all these models follows from uniqueness of a hidden or internal system. While this is expected to hold very generally, results that are not restricted to particular models appear only for discrete spin systems [9]. The present paper now narrows the gap. It provides a proof of the preservation of the Gibbs property of the time-evolved Gibbs-measures of a general continuous spin system under site-wise independent dynamics, for short times, even when the initial measure is in the strong coupling regime. More generally than for time-evolution, we prove our results directly for general two layer systems, consisting of (1) a Gibbs-measure in the first layer, that is (2) subjected to local transition kernels mapping the first layer variables to second layer variables. This generalizes the notion of a hidden Markov model where the second layer plays the role of a noisy observation. Such models have motivation in a variety of fields. Let us mention for example that they appear in biology as models of gene regulatory networks where the vertices of the network are genes and the variables model gene expression activity.

A measure is a Gibbs measure when the single-site conditional probabilities depend on the conditioning in an essentially local way. Our main statement (Theorem 2.6) is an explicit upper bound on the continuity of the single-site conditional probabilities of the second layer system as a function of the conditioning. This is valid when the transition kernels don't fluctuate too much, even when the first layer system is in a strong coupling regime. Our result holds for discrete or continuous compact state spaces and general interactions and is based on Dobrushin uniqueness. To formulate the resulting continuity estimate for the conditional probabilities we don't need any a priori metric structure on the local spin spaces: The natural metric on the second layer single spin space is created by the variational distance between the a-priori measures in the first layer that are obtained by conditioning on second layer configurations(see Theorem 2.6).

On the way to this result, we exhibit a simple criterion for Dobrushin uniqueness for Gibbs-measures (of one layer). It is easy to check and can be of use beyond the study of (non)-Gibbsianness.

Intuitively, it demands that the sum over the interaction terms in the Hamiltonian coupling the sites i and j should not fluctuate too much when it is viewed as a random variable at the site i under the corresponding local a-priori measure (see Definition 2.1). So even when one has a large interaction, better concentration properties of the a priori measures can still imply an overall small Dobrushin constant. This is a generalization

of the simple large-field criterion ensuring Dobrushin-uniqueness in the Ising model (see p.147 example 8.13 of [1] and [20]) to general spaces (Theorem 2.2).

In Theorem 2.4 we state as a corollary that "concentration implies concentration". By this we mean that there are Gaussian concentration inequalities for functions of the coupled system, with explicit decay rate (even when there is strong coupling) if the a priori measures concentrate well enough.

The criterion we need for the study of the second layer model is based on the description of the interplay between the possible largeness of the initial interaction and the strength of the coupling to the second layer found in Theorem 2.2 (when the initial apriori measures are replaced with conditional apriori measures). To ensure Gibbsianness of the second layer model, we thus need small fluctuations of the initial Hamiltonian w.r.t. the a-priori measures in the first layer that are obtained by conditioning on second layer configurations. The estimates on the spatial memory of the single-site second layer conditional probabilities follow naturally by evoking Dobrushin-uniqueness estimates on comparison of the Gibbs-measures with perturbed specifications and chain-rule type of arguments.

To illustrate the simplicity of our approach to get explicit estimates on the spatial decay we prove short-time Gibbsianness of (Heisenberg)-model of $(q-1)$-dimensional rotators for general $q \geq 2$ under diffusive time-evolution on the $(q-1)$-spheres, and provide an explicit estimate on the time-interval for which the time-evolved measure stays Gibbs. This will be supplemented by arguments that are more specific to the rotators which give us precise continuity estimates in terms of the Euclidean distances on the spheres.

In Section 2 we formulate our main results. In Section 3 we provide the proofs of Theorem 2.2 and 2.4, in Section 4 we provide the proof of Theorem 2.6, and in Section 5 we provide the proofs of Theorem 2.7 and Proposition 2.8 and provide some related results. We also give the proof of Theorem 2.9 in Section 5.

2 Main Results

2.1 A criterion for Dobrushin uniqueness for concentrated a priori measures

Let G be a countable vertex set, and assume that $\sigma = (\sigma_i)_{i\in G}$ are spin-variables taking values in a measurable (standard Borel) space S (single-spin space). In our general setup we don't need to make a metric structure on S explicit. We further denote by $\Omega = S^G$ the configuration space of our system equiped with the Borel σ-algebra.

Let γ be the Gibbs specification(collection of finite-volume conditional distributions that depend in a continuous way on the conditioning) for a given interaction potential $\Phi = (\Phi_A)_{A\subset G}$ (where $\Phi_A : S^G \mapsto \mathbb{R}$ are functions that depend only on the spin-variables in A for finite subsets A of G) and a priori probability measure α on the single-site spaces, i.e for any finite $\Lambda\subset G$ and $\bar\sigma \in S^G$ we define $\gamma_\Lambda(\cdot|\bar\sigma) \in \gamma$ as

$$\gamma_\Lambda(d\sigma_\Lambda|\bar\sigma) := \exp\bigl(-\sum_{A\cap\Lambda\neq\emptyset} \Phi_A(\sigma_\Lambda\bar\sigma_{G\setminus\Lambda})\bigr)\prod_{i\in\Lambda}\alpha(d\sigma_i)/Z_\Lambda(\bar\sigma) \tag{1}$$

with the normalization constant $Z_\Lambda(\bar\sigma)$.

We assume the summability property

$$|||\Phi||| := \sup_{i\in G}\sum_{A\ni i}|A|\|\Phi_A\|_\infty < \infty \tag{2}$$

for the interaction Φ. In the sequel we will always write i for $\{i\}$, i^c for $G\setminus\{i\}$ and Λ^c for $G\setminus\Lambda$. We further denote by $C=(C_{ij})_{i,j\in G}$ the *Dobrushin interdependence matrix*, with entries given by

$$C_{ij} = \sup_{\zeta,\eta\in\Omega;\ \zeta_{j^c}=\eta_{j^c}} \|\gamma_i(\cdot|\zeta)-\gamma_i(\cdot|\eta)\|. \tag{3}$$

where $\|\nu_1-\nu_2\| := \sup_{f;|f|\le 1}|\nu_1(f)-\nu_2(f)| = \frac{1}{2}\lambda(|h_1-h_2|)$ whenever ν_1 and ν_2 are probability measures that are absolutely continuous with respect to the measure λ with λ-densities h_1 and h_2 respectively (i.e. $\|\nu_1-\nu_2\|$ is one half of the variational distance between ν_1 and ν_2). The corresponding *Dobrushin constant* is also given as

$$c := \sup_{i\in G}\sum_{j\in G} C_{ij}.$$

and we recall that whenever $c<1$ (Dobrushin uniqueness condition) then γ admits at most one Gibbs measure [14, 1]. It is known that for a potential Φ satisfying (2) there is a sufficiently small β such that $\beta\Phi$ satisfies Dobrushin uniqueness and the measure is in a small coupling regime. We will prove Dobrushin uniqueness for a potential with possibly very large (but finite) (2) when the measure α concentrates. In fact, we can also deduce Dobrushin uniqueness for weak coupling from the bound we will provide on the Dobrushin's contant c.

For our purposes we employ the following definition.

Definition 2.1 *For a function $F: S^G \mapsto \mathbb{R}$ we define the* $\alpha; i, j$**-deviation** $dev_{\alpha;i,j}$ *of F to be*

$$dev_{\alpha;i,j}(F) := \sup_{\substack{\zeta,\eta\in S^G\\ \zeta_{j^c}=\eta_{j^c}}}\inf_B \int \alpha(d\sigma_i)\Big|F(\sigma_i\eta_{i^c}) - F(\sigma_i\zeta_{i^c}) - B\Big|. \tag{4}$$

This quantity is the worst-case linear deviation of the variation of F at the site j viewed as a random variable w.r.t. to σ_i under $\alpha(d\sigma_i)$. Note that clearly the deviation is bounded by $\delta_j(F)$ the jth *oscillation* of F , i.e.
$\mathrm{dev}_{\alpha;i,j}(F) \le \delta_j(F) = \sup_{\substack{\zeta,\eta\in S^G\\ \zeta_{j^c}:=\eta_{j^c}}}\Big|F(\eta)-F(\zeta)\Big|.$

Then our first result is as follows.

Theorem 2.2 *The Dobrushin constant c is bounded by*

$$c \le \sup_{i\in G}\sum_{j\in G}\exp\Big(\sum_{A\supset\{i,j\}}\delta(\Phi_A)\Big)\, dev_{\alpha;i,j}(H_i) \tag{5}$$

where $\delta(\Phi_A)$ is the oscillation of Φ_A defined as $\delta(\Phi_A) := \sup_{w,\bar{w}\in S^G}|\Phi_A(w)-\Phi_A(\bar{w})|$ and $H_V := \sum_{A\cap V\neq\emptyset}\Phi_A$

The use of this criterion lies in the fact that, even when the interaction potential is large, $\mathrm{dev}_{\alpha;i,j}(H_i)$ can be small, when α is close to a Dirac measure. A simple example for this to happen is an Ising model at large external field. As a less trivial application of the criterion to a spin-model where S is not discrete we discuss the Gauss-Weierstrass kernel in the rotator example of Section 2.4 where we prove short-time Gibbsianness.

Of course, when the potential is small to begin with, the r.h.s. of (5) will be small, independently of α, so the theorem can be used for both strong couplings and concentrated a priori-measures and weak coupling.

2.2 Concentration implies concentration

Dobrushin uniqueness implies also the existence of a Gibbs measure μ (if the local spin space S is standard Borel (Theorem 8.7. [1]).) This unique measure μ then has further nice properties; e.g. general Gaussian estimates on the concentration of an observable F around its mean hold [6, 7]. We believe that the concentration result below is worth mentioning.

The estimate on the Dobrushin matrix C that leads to the upper bound (5) on c then also implies the Gaussian concentration estimate which we will give in Theorem 2.4.

Definition 2.3 *We call the matrix B with entries*

$$B_{ij} := dev_{\alpha;i,j}(H_i). \tag{6}$$

the deviation matrix of the potential Φ w.r.t. α.

To formulate the concentration theorem let us write $||B||_1 := \sup_{j\in G}\sum_{i\in G} B_{i,j}$ and $||B||_\infty := \sup_{i\in G}\sum_{j\in G} B_{i,j}$ for the corresponding matrix-norms.

Theorem 2.4 *Suppose $||B||_1, ||B||_\infty < \frac{1}{s}$ where $s := \exp\Big(\sup_{i\neq j}\sum_{A\supset\{i,j\}}\delta(\Phi_A)\Big)$. Then for any bounded measurable function $F(\sigma)$ and $\forall r \geq 0$ holds the inequality*

$$\mu\Big(F(\sigma) - \mu\big(F(\sigma)\big) \geq r\Big) \leq \exp\left(-\Big(1 - s\|B\|_\infty\Big)\Big(1 - s\|B\|_1\Big)\frac{r^2}{2\big\|\underline{\delta}(F)\big\|_{l^2}^2}\right) \tag{7}$$

Here we have written $\big\|\underline{\delta}(F)\big\|_{l^2}^2 \equiv \sum_{i\in G}(\delta_i(F))^2$.

2.3 Two-layer models - Goodness of Gibbsianness

Let us now formulate our assumptions on a two-layer system over a graph G. To each vertex will be associated two local state spaces. A particular example will be given by the site-wise independent time-evolution of Section 2.4. So, in general let S and additionally S' be measurable (standard Borel) spaces. This implies in particular existence of all regular conditional probabilities. Again, no a priori metric will be used explicitly. We refer to S as the initial (first layer) spin space and to S' as the image (second layer) spin space. Let the *joint a priori measure* $K(d\sigma_i, d\eta_i)$ be a Borel probability measure on the product space $S \times S'$. We assume non-nullness of K (positivity of measure for all open sets). We assume further that K can be written in the form $K(d\sigma_i, d\eta_i) =$

$k(\sigma_i, \eta_i)\alpha(d\sigma_i)\alpha'(d\eta_i)$ where $\alpha(d\sigma_i) \equiv \int_{S'} K(d\sigma_i, d\eta_i)$ and $\alpha'(d\eta_i) \equiv \int_S K(d\sigma_i, d\eta_i)$ with $k > 0$.

Our initial model (probability measure on S^G) is by definition a Gibbs distribution for the specification given in terms of the potential Φ according to (1) where we now put as an a priori measure the marginal of K on the first layer, that is $\alpha(d\sigma_i) \equiv \int_{S'} K(d\sigma_i, d\eta_i)$. It is important to note that we don't assume uniqueness of the Gibbs measure for this specification. In practice α might be given beforehand and K is then obtained by specifying a transition kernel $K(d\eta_i|\sigma_i)$ from the first layer to the second layer. We will always denote by $\sigma_i \in S$ the local variable (spin) for the initial model and $\eta_i \in S'$ the local variable (spin) for the image model.

Let $\mu(d\sigma)$ be a Gibbs measure for the first layer for potential Φ and a priori measure α. Our aim is then: Study the conditional probabilities of the second layer measure defined by

$$\mu'(d\eta) := \int_{S^G} \mu(d\sigma) \prod_{i \in G} K(d\eta_i|\sigma_i).$$

This form appears for instance in the study of a stochastic time evolution, starting from an initial measure μ where the kernel $K(d\eta_i|\sigma_i)$ will be dependent on time and is applied independently over the spins (infinite-temperature dynamics). In case studies it has been observed that the map $\mu \mapsto \mu'$ may create an image measure that is not a Gibbs measure anymore. On the other hand, in all examples observed, Gibbsianness was preserved at short times where K_t is a small perturbation of $\delta_{\eta_i}(d\sigma_i)$. We aim here to give a criterion that implies this in all generality, not using any specifics of the model but only the relevant underlying structure. In particular we are not restricting ourselves to discrete spin spaces.

Our main result Theorem 2.6 is a criterion for the Gibbs property of the second layer measure that is easily formulated and verified in concrete examples. Moreover, we give explicit bounds on the dependence of the conditional probabilities of the second layer measure on the variation of the conditioning.

We said that we will not use any a priori metric on the spaces S and S'; indeed the natural metric that shall be used for continuity in this setup shall be given by the variational distance of the conditional a priori measures in the first layer, conditional on the second layer.

Definition 2.5 *We call*

$$d'(\eta_j, \eta_j') := \|\alpha_{\eta_j} - \alpha_{\bar{\eta}_j}\|$$

the posterior (pseudo-)metric associated to K on the second layer space.

Here $\alpha_{\eta_i}(d\sigma_i) = K(d\sigma_i|\eta_i)$ are the a priori measures in the first layer that are obtained by conditioning on second layer configurations.

In the language of statistics, α_{η_i} is the "posterior measure" depending on the observation η_i in the second layer single spin space. Stated abstractly, the metric d' is the pull back-metric of the map $\eta_i \mapsto \alpha_{\eta_i}(d\sigma_i)$ from single-site configurations in the second layer to single-site measures in the first layer. While this metric seems to be non-explicit, we will show in the rotator example how it can be estimated in terms of a more familiar metric (Euclidean metric).

It is well-known that an investigation of the Gibbs property of the second layer measure must be based on an analysis of the first layer conditional on configurations in the second layer [4, 5, 8]. So, our estimates will naturally contain quantities that reflect this aspect. The main ingredient will be a matrix $\bar{B}$ that is a uniform bound (over possible conditionings) on the conditional deviation matrix $B(\eta)$ of the first layer system. More precisely, let us put

$$\begin{aligned}\bar{C}_{ij} &:= \exp\Big(\sum_{A\supset\{i,j\}}\delta(\Phi_A)\Big)\bar{B}_{ij} \qquad \text{where} \\ \bar{B}_{ij} &:= \sup_{\eta_i}\mathrm{dev}_{\alpha_{\eta_i};i,j}(H_i).\end{aligned} \tag{8}$$

We warn the reader not to confuse $\mathrm{dev}_{\alpha_{\eta_i};i,j}(H_i)$ with $\mathrm{dev}_{\alpha;i,j}(H_i)$. While the second quantity may be big and correspondingly the unconstrained first layer system in a non-uniqueness regime, the first one might still be small and correspondingly the constrained layer system in a uniqueness regime. This is e.g. the case for a time-evolution started at low temperature, for small times. We denote by γ' the class of all finite-volume conditional distributions of the transfored model with full η-conditioning. Then we have the following theorem.

Theorem 2.6 *Suppose that the first layer system has an infinite-volume Gibbs measure $\mu = \lim_n \mu^{\bar{\sigma}}_{\Lambda_n}$ obtained for a boundary condition $\bar{\sigma}$ and along a suitable sequence of volumes Λ_n.*

Suppose further that $\sup_i \sum_j \bar{C}_{ij} < 1$.

1. *Then γ' is a specification and the second layer measure μ' is a Gibbs measure for the specification γ'.*

2. *γ' satisfies the continuity estimate*

$$\Big\|\gamma'_i(d\eta_i|\eta_{i^c}) - \gamma'_i(d\eta_i|\bar{\eta}_{i^c})\Big\| \le \sum_{j\in G\setminus i} Q_{i,j}\,d'(\eta_j,\bar{\eta}_j). \tag{9}$$

where

$$Q_{i,j} = 4e^{2\sum_{A\ni i}\|\Phi_A\|_\infty}\Big(\sum_{k\in G\setminus i}\delta_k\Big(\sum_{A\supset\{i,k\}}\Phi_A\Big)\bar{D}_{kj}\Big)e^{\sum_{A\ni j}\delta_j(\Phi_A))} \tag{10}$$

with $\bar{D} = \sum_{n=0}^\infty \bar{C}^n$.

Note that the first layer system may be very well in a phase transition regime. For arbitrarily large interactions Φ, good concentration of the conditional measures α_{η_i} can still lead to a small "Dobrushin matrix" $\bar{C}$, when the deviation matrix $\bar{B}(\eta)$ is uniformly small in η. In short: Uniform conditional Dobrushin uniqueness of the first layer implies Gibbsianness of the second layer, with explicit estimates.

The matrix Q describing the spatial loss of memory of the variation of the conditioning, depends on the summability properties of the potential Φ and the decay of the "Dobrushin-matrix" $\bar{C}$. Note that the summability property we impose on the initial potential (2) implies the finiteness of (10). In particular we have the following bound on the entries of the Q-matrix;

$$Q_{ij} \le 4\exp\Big(4\sup_{i\in G}\sum_{A\ni i}||\Phi_A||_\infty\Big)\big(M\bar{D}\big)_{ij}, \tag{11}$$

where M is the matrix given by $M_{ik} = \begin{cases} \delta_k\left(\sum_{A\supset\{i,k\}} \Phi_A\right) & \text{if } i \neq k; \\ 0 & \text{if } i = k. \end{cases}$

All these quantities are easily made explicit in examples.

2.4 Goodness of short-time Gibbsianness for time-evolved rotator models

Let us get more concrete. Consider the rotator model on G, with both first layer and second layer local spin spaces equal to S^{q-1}, the sphere in q-dimensional Euclidean space, with $q \geq 2$.

Take as a Hamiltonian of the first layer system in infinite volume

$$H(\sigma) = -\sum_{i,j\in G} J_{ij}\sigma_i \cdot \sigma_j \tag{12}$$

with $\sup_i \sum_j |J_{ij}| < \infty$ where we assume that $J_{ii} = 0$ for each $i \in G$. Let K be given by $K(d\sigma_i, d\eta_i) = K_t(d\sigma_i, d\eta_i) = k_t(\sigma_i, \eta_i)\alpha_0(d\sigma_i)\alpha_0(d\eta_i)$, where α_0 is the equidistribution on S^{q-1} and k_t is the heat kernel on the sphere, i.e.

$$\left(e^{\Delta t}\varphi\right)(\eta_i) = \int \alpha_o(d\sigma_i)k_t(\sigma_i, \eta_i)\varphi(\sigma_i), \tag{13}$$

where Δ is the Laplace-Beltrami operator on the sphere and φ is any test function. k_t is also called the **Gauss-Weierstrass kernel**. The time-evolved measure is given by

$$\mu_t(d\eta) = \int \mu(d\sigma)\prod_i k_t(\sigma_i, \eta_i)\alpha_0(d\eta_i). \tag{14}$$

It has the product over the equidistributions on the spheres as an infinite-time local limiting measure

$$\lim_{t\uparrow\infty} \mu_t(d\eta) = \bigotimes_{i\in G} \alpha_0(d\eta_i). \tag{15}$$

Denote γ'_t by the class of all finite-volume conditional distributions of the time-evolved measure with full η-conditioning. Then the following continuity estimates on the conditional probabilities of the time-evolved model hold.

Theorem 2.7 *Denote by $d(\eta, \eta')$ the induced metric on the sphere S^{q-1} (with $q \geq 2$) obtained by embedding the sphere into the Euclidean space $\mathbb{R}^q$.*

Assume that

$$\sqrt{2}\left(\sup_i \sum_{j\in G} e^{|J_{ij}|}|J_{ij}|\right)\left(1 - e^{-(q-1)t}\right)^{\frac{1}{2}} < 1. \tag{16}$$

Then the following holds.

1. *The measure μ_t is Gibbs for a specification γ'_t, and*

2. *γ_t satisfies the continuity estimate*

$$\left\|\gamma'_{i,t}(d\eta_i|\eta_{i^c}) - \gamma'_{i,t}(d\eta_i|\bar{\eta}_{i^c})\right\| \leq \sum_{j\in G\setminus i} \bar{Q}_{i,j}(t)d(\eta_j, \bar{\eta}_j), \tag{17}$$

with

$$\bar{Q}_{i,j}(t) := \frac{1}{2}\min\Big\{\sqrt{\frac{\pi}{t}}Q_{i,j}(t), e^{4\sum_l |J_{jl}|} - 1\Big\} \tag{18}$$

where

$$Q_{i,j}(t) = 8e^{4\sup_{i\in G}\sum_{j\in G}|J_{ij}|}\sum_{k\in G\setminus i}|J_{ik}|\bar{D}_{kj}(t), \tag{19}$$

$\bar{D}(t) = \mathbf{1} + \sum_{n=1}^{\infty}\left(1 - e^{-(q-1)t}\right)^{\frac{n}{2}}A^n$, A is the matrix whose entries are given by $A_{ij} = e^{|J_{ij}|}|J_{ij}|$ and $\mathbf{1}$ is the identity matrix.

In the definition of Q_{ij} in the above theorem we have used the bound (11) on the Q_{ij} in Theorem 2.6.

The proof of the theorem follows from three ingredients: 1) Theorem 2.6 which gives a continuity estimate in terms of the posterior metric d', 2) a comparison result between d' and d, see Proposition 2.8 and 3) a telescoping argument over sites in the conditioning.

It is straightforward to apply Theorem 2.6 to our model and obtain a result formulated in d'. However, a more natural metric we would prefer to use is d, and so we should use a comparison argument, applying Proposition 2.8. What continuity estimates do we expect to gain from this? It is elementary to see that for the initial kernel

$$\Big\|\gamma_{t=0}(d\eta_i|\eta_{i^c}) - \gamma_{t=0}(d\eta_i|\bar{\eta}_{i^c})\Big\| \leq e^{2\sum_{j\in G}|J_{ij}|}\sum_{j\in G}|J_{ij}|d(\eta_j, \bar{\eta}_j) \tag{20}$$

We see that continuity can be measured in terms of d, due to the Lipschitz property of the initial Hamiltonian, and the spatial decay is provided by the decay of the couplings.

So, at small time t, we are aiming at a similar continuity estimate to hold which is uniform in t as t goes to zero. Now, while estimating d' against d we have accumulated a nasty factor $\frac{1}{\sqrt{t}}$ that blows up when time t goes to zero. We note that this is not just an artefact of Proposition 2.8, but the posterior metric between two points on the sphere indeed blows up like $\frac{1}{\sqrt{t}}$, as can be seen from the proof. At first sight this does not seem to be a problem in the definition of $Q_{ij}(t)$ because the off-diagonal entries of the matrix $\bar{D}_{ij}(t)$ are suppressed by the same factor proportional to $\sqrt{t}$ that appears in (16). This suppression follows from a bound on the corresponding Dobrushin matrix of this order. Unfortunately the diagonal terms of $\bar{D}(t)$ give rise to blow-up for sites i and j that are within the range of the potential. As it is clear from the proof, this blow-up is understandable since so far we did not employ any continuity properties of the initial Hamiltonian w.r.t. the Euclidean metric. Without further conditions of this sort clearly no continuity can be expected, as even a system of two sites with the Hamiltonian being a step function shows.

Now, to disentangle these *local effects* from the *global effects* treated so far, we use in the third step a telescoping argument over the conditioning. Exploiting Lipschitz-continuity w.r.t. a single argument of the Hamiltonian we obtain the second term in the minimum in (18) which puts a time-independent ceiling to the blow-up for small times. This solves the blow-up problem.

In this context let us also exhibit the comparison estimate of the two metrics d' and d that we also deem of interest in itself.

Proposition 2.8 *There is an estimate of the posterior metric d' associated to the measure K_t of the form*

$$d'(\eta_j, \bar{\eta}_j) \le F_{q,t}\big(d(\eta_j, \bar{\eta}_j)\big). \tag{21}$$

The function $F_{q,t}$ satisfies the following:

1. *For any $q \ge 2$, $x \in [0, 2]$ and $t > 0$ we have the estimate*

$$F_{q,t}(x) \le 4P\Big(0 \le G \le \frac{\arcsin \frac{x}{2}}{\sqrt{2t}}\Big) \le \frac{\sqrt{\pi}x}{2\sqrt{t}} \tag{22}$$

   where G is a standard normal variable.

2. *In general dimensions $q \ge 2$ more information can be derived by the expansion*

$$\begin{aligned} F_{q,t}(x) &= \sum_{m=0}^{\infty} a_{q,m}(t) P_{2m+1}\Big(q, \frac{x}{2}\Big) \text{ with} \\ a_{q,m}(t) &= e^{-(2m+1)(2m+q-1)t} \frac{(-1)^m 4N(q,m)\Gamma\left(\frac{q}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{q-1}{2}\right)} \prod_{i=0}^{m} \left(\frac{2i-1}{q+2i-1}\right) \end{aligned} \tag{23}$$

   in terms of Legendre polynomials $P_n(q, s)$ of degree n in dimension q (see the definition 5.3)and $N(q, m)$ is also the dimension of the space of spherical harmonics of degree n in dimension q (see (94)).

Remark: The proof uses a coupling argument and a reflection principle for diffusions on the sphere under reflection at the equator.

2.5 Goodness of Gibbsianness for local approximations

As another consequence from the general theorem we prove that any sufficiently fine local coarse graining preserves the Gibbs property. Here the fineness of the coarse graining has to be compared relative to the scale in the local state spaces on which the initial Hamiltonian is varying.

We thus need a bit more structure, namely let (S, d) now be a metric space. Let a decomposition be given of the form $S = \bigcup_{s' \in S'} S_{s'}$. Here S' may be a finite or infinite set. Put $T(s) := s'$ for $S_{s'} \ni s$. This defines a deterministic transformation on S, called the *fuzzy map*. With this map we associate to each $s' \in S'$ a corresponding a priori measure on $S_{s'}$ (say $\alpha_{s'}$). Note that here $\alpha_{s'}$ is the corresponding analogue of $\alpha_\eta = K(\cdot|\eta)$ for the fuzzy map.

Theorem 2.9 *Assume the Lipschitz-property for the j-variation of the initial Hamiltonian*

$$\sup_{\substack{\zeta, \bar{\zeta} \\ \zeta_{j^c} = \bar{\zeta}_{j^c}}} \Big| H_i(\sigma_i \zeta_{i^c}) - H_i(\sigma_i \bar{\zeta}_{i^c}) - \Big(H_i(a_i \zeta_{i^c}) - H_i(a_i \bar{\zeta}_{i^c})\Big) \Big| \le L_{ij} d(\sigma_i, a_i). \tag{24}$$

Suppose that

$$\frac{\rho}{2} \sup_{i \in G} \sum_{j \in G} \exp\Big(\frac{1}{2} \sum_{A \supset \{i,j\}} \delta(\Phi_A)\Big) L_{ij} < 1$$

where $\rho = \sup_{s'} diam(S_{s'})$ denotes the fineness of the decomposition.

1. *Then, for any initial Gibbs measure μ of the specification Φ with an arbitray a priori measure α the transformed measure $T(\mu)$ is Gibbs for a specification γ'.*

2. *The entries C'_{ij} of the Dobrushin interdependence matrix of γ' are bounded by Q_{ij} given by (10) where we have to put*

$$\bar{C}_{ij} = \frac{\rho}{2} \exp\Big(\frac{1}{2} \sum_{A \supset \{i,j\}} \delta(\Phi_A)\Big) L_{ij}.$$

Answering a question of Aernout van Enter, this provides a class of examples where S and S' are different (one may be continuous, the other not), the initial measure may be in the phase transition regime, and the image measure will be Gibbs. To think of an even more concrete example, let take the rotor-model (12). Divide the sphere $S^{q-1} = \bigcup_{s'} S_{s'}$ into "countries" $S_{s'}$. Then the correspondingly discretized model on the country-level is still Gibbs whenever there is no country with diameter bigger then $\Big(\sup_i \sum_{j \in G} e^{|J_{ij}|}|J_{ij}|\Big)^{-1}$.

As a concluding remark let us mention that we may very well apply our method also to other well-known examples of transforms of Gibbs measures that may potentially lead to renormalization group pathologies. For instance, also the decimation transformation mapping a Gibbs measure on the lattice to its restriction to a sublattice can be cast in this framework. Theorem 2.6 then implies the statement that the projected measure is always Gibbs if the interaction is sufficiently small in triple norm. The posterior metric for configurations on the projected lattice then becomes the discrete metric $d'(\eta_i, \eta'_i) = 1_{\eta_i \neq \eta'_i}$ and hence the matrix element Q_{ij} becomes a bound on the Dobrushin interdependence matrix of the image system.

3 On the proofs on Theorem 2.2 and 2.4:

In this section we provide proofs of Theorem 2.2 and 2.4 and also state and prove some related results. We start with the

Proof of Theorem 2.2: The idea of the proof is to find an estimate on the Dobrushin interdependence matrix as in the proof of Proposition 8.8 of [1]. This involves estimating the variation of the single-site measure at a given site $i \in G$ when varying the boundary condition at some site $j \in G \backslash i$. That is we fix $\zeta, \eta \in \Omega$ with $\zeta_{j^c} = \eta_{j^c}$ and put $u_0(\sigma_i) = -H_i(\sigma_i \zeta_{i^c})$ and $u_1(\sigma_i) = -H_i(\sigma_i \eta_{i^c})$. We proceed further by taking the linear interpolation $u_t = tu_1 + (1-t)u_0$ of u_1 and u_0. It follow from this linear interpolation that

$$\delta(u_t) \leq \sum_{A \supset \{i,j\}} \delta(\Phi_A). \tag{25}$$

Setting $h_t = e^{u_t}/\alpha(e^{u_t})$ and $\lambda_t(d\sigma_i) = h_t(\sigma_i)\alpha(d\sigma_i)$ we note that $\lambda_0(d\sigma_i) = \gamma_i(d\sigma_i|\zeta)$ and $\lambda_1(d\sigma_i) = \gamma_i(d\sigma_i|\eta)$. We now observe that

$$\begin{aligned} 2\|\lambda_0 - \lambda_1\| &= \int \alpha(d\sigma_i)|h_1(\sigma_i) - h_0(\sigma_i)| = \int \alpha(d\sigma_i)\Big|\int_0^1 dt \frac{d}{dt} h_t(\sigma_i)\Big| \\ &\leq \int_0^1 dt\ \lambda_t \Big| H_i(\cdot\zeta_{i^c}) - H_i(\cdot\eta_{i^c}) - \lambda_t\Big(H_i(\cdot\zeta_{i^c}) - H_i(\cdot\eta_{i^c})\Big)\Big| \\ &\leq 2\int_0^1 dt \exp\Big(\delta(u_t)\Big)\inf_B \int \alpha(d\sigma_i)\Big|H_i(\sigma_i\zeta_{i^c}) - H_i(\sigma_i\eta_{i^c}) - B\Big|. \end{aligned} \tag{26}$$

It follows from(3),(4) and (25) that

$$C_{ij} \leq \exp\Big(\sum_{A\supset\{i,j\}} \delta(\Phi_A)\Big)\mathrm{dev}_{\alpha;i,j}(H_i).$$

The rest of the proof follows from the definition of the Dobrushin constant c.

□

Sometimes it is useful to use quadratic variation instead of the linear variation $\mathrm{dev}_{\alpha;i,j}$ to obtain an explicit bound, as we shall see in the proof of Theorem 2.7 below. We define this quadratic variation as follows.

Definition 3.1 *For any bounded measurable function F on Ω we define for any pair $i,j \in G$ $std_{\alpha;i,j}(F)$ as*

$$std_{\alpha;i,j}(F) := \sup_{\zeta,\bar\zeta\in\Omega,\zeta_{j^c}=\bar\zeta_{j^c}} \inf_B \left(\int d\alpha(d\sigma_i)\Big(F(\sigma_i\zeta_i^c) - F(\sigma_i\bar\zeta_i^c) - B\Big)^2\right)^{\frac{1}{2}}. \tag{27}$$

The quantity $\mathrm{std}_{\alpha;i,j}(F)$ is the worst-case quadratic deviation of the variation of F at the site j viewed as a random variable w.r.t. to σ_i under $\alpha(d\sigma_i)$. Clearly $\mathrm{dev}_{\alpha;i,j}(F) \leq \mathrm{std}_{\alpha;i,j}(F)$, so we could bound the inequality in Theorem 2.2 in terms of the quadratic variation; going directly into the proof however gives a slightly better constant.

This gives rise to the following "quadratic version" of Theorem 2.2.

Proposition 3.2 *The Dobrushin constant c is also bounded by*

$$c \leq \frac{1}{2}\sup_{i\in G}\sum_{j\in G}\exp\Big(\frac{1}{2}\sum_{A\supset\{i,j\}}\delta(\Phi_A)\Big)std_{\alpha;i,j}(H_i). \tag{28}$$

Proof: The proof uses the same arguments employed in the proof of Theorem 2.2 above, the only difference being that we have a quadratic estimate (resulting from the Cauchy-Schwartz inequality) in

$$\begin{aligned} 2\|\lambda_0 - \lambda_1\| &= \int \alpha(d\sigma_i)|h_1(\sigma_i) - h_0(\sigma_i)| = \int \alpha(d\sigma_i)\Big|\int_0^1 dt \frac{d}{dt} h_t(\sigma_i)\Big| \\ &\leq \int_0^1 dt\ \lambda_t \Big| H_i(\cdot\zeta_{i^c}) - H_i(\cdot\eta_{i^c}) - \lambda_t\Big(H_i(\cdot\zeta_{i^c}) - H_i(\cdot\eta_{i^c})\Big)\Big| \\ &\leq \int_0^1 dt \exp\Big(\frac{\delta(u_t)}{2}\Big)\inf_B\Big(\int \alpha(d\sigma_i)\Big(H_i(\sigma_i\zeta_{i^c}) - H_i(\sigma_i\eta_{i^c}) - B\Big)^2\Big)^{\frac{1}{2}}. \end{aligned} \tag{29}$$

$\square$

If the initial Hamiltonian satisfies a Lipschitz-property w.r.t. a given metric d on the local state space an estimate of the Dobrushin constant can be formulated as follows.

Corollary 3.3 *Suppose the Lipschitz-condition (24). Then we have*

$$c \leq \frac{1}{2} \sup_{i \in G} \sum_{j \in G} \exp\Big(\frac{1}{2} \sum_{A \supset \{i,j\}} \delta(\Phi_A)\Big) L_{ij} \inf_{a_i \in S} \Big(\int d^2(\sigma_i, a_i) \alpha(d\sigma_i) \Big)^{\frac{1}{2}}.$$

The proof of the corollary follows from the last inequality in (29), since taking the infimum over B is less than or equal to taking the infimum over $a_i \in S$ when we substitute B with $H_i(a_i \zeta_{i^c}) - H_i(a_i \eta_{i^c})$.

A somewhat more abstract reformulation of the bounds on the Dobrushin's constant can be given in terms of appropriately defined norms of the potential.

Corollary 3.4 *Define for Φ the norms*

$$\begin{aligned} |||\Phi|||_{dev_\alpha} &:= \sup_{i \in G} \sum_{j \in G \setminus \{i\}} \sum_{A \supset \{i,j\}} dev_{\alpha;i,j}(\Phi_A) \\ |||\Phi|||_{std_\alpha} &:= \sup_{i \in G} \sum_{j \in G \setminus \{i\}} \sum_{A \supset \{i,j\}} std_{\alpha;i,j}(\Phi_A). \end{aligned} \tag{30}$$

Then we have for $|||\Phi||| < \infty$ that the Dobrushin constant c of the specification for Φ satisfies the following bounds

$$c \leq e^{2|||\Phi|||} |||\Phi|||_{dev_\alpha} \quad \textit{and} \quad c \leq \frac{1}{2} e^{|||\Phi|||} |||\Phi|||_{Std_\alpha}. \tag{31}$$

Once the definitions are made the proof is obvious. Note further that c is finite as long as $|||\Phi|||$ is because $|||\Phi|||_{std_\alpha} \leq 2|||\Phi|||$. We finally give the proof of the bounds in the "Concentration implies concentration"-theorem.

Proof of Theorem 2.4: Note that the hypothesis $||B||_\infty < \frac{1}{s}$ (as we will see below) implies that we are in the uniqueness regime. Then for any bounded measurable function F on Ω it follows from Theorem 1 of [6] that under the unique Gibbs measure μ

$$\mu\Big(F - \mu(F) \geq r\Big) \leq \exp\left(-\frac{r^2}{2} \frac{(1-c)(1-c_t)}{||\underline{\delta}(F)||_{l_2}^2} \right) \quad \forall\, r \geq 0, \tag{32}$$

where c and c_t are respectively the Dobrushin constants of the Dobrushin interdependence matrix and its transpose. It follows from the definitions of c and c_t and the bound in Theorem 2.2 that

$$\begin{aligned} c &\leq \exp\Big(\sup_{i \neq j} \sum_{A \supset \{i,j\}} \delta(\Phi_A)\Big) \sup_{i \in G} \sum_{j \in G} \mathrm{dev}_{\alpha;i,j}(H_i) = s||B||_\infty \\ c_t &\leq \exp\Big(\sup_{i \neq j} \sum_{A \supset \{i,j\}} \delta(\Phi_A)\Big) \sup_{j \in G} \sum_{i \in G} \mathrm{dev}_{\alpha;i,j}(H_i) = s||B||_1. \end{aligned}$$

$\square$

Note that the validity of Theorem 2.4 depends on c and c_t being less than one. In our criterion (5) the smallness of the $\mathrm{dev}_{\alpha;i,j}$'s is the main ingredient for c and c_t to be less than one. This smallness of the $\mathrm{dev}_{\alpha;i,j}$'s is caused by good "concentration" properties of α even if the interaction is strong and possibly by the weakness of the interaction.

4 On the proof Theorem 2.6 and related results

The purpose of this section is to give the proof of Theorem 2.6 outlined in Section 2.3 of the introduction. The main ingredient to the proof is to show the lack of phase transitions in some intermediate system and exploit the consequences for decay of spatial memory. Recall from Section 2.3 that our initial system was given by the Gibbs measure μ admitted by the specification γ obtained from the interaction Φ and an a priori measure $\alpha = \int K(\cdot, d\eta_i)$ described above. Thus for a given boundary condition $\bar{\sigma}$ and any finite volume $\Lambda \subset G$ we write $\gamma_\Lambda(\cdot|\bar{\sigma}) \in \gamma$ as

$$\gamma_\Lambda(d\sigma_\Lambda|\bar{\sigma}) = \frac{\exp\Big(-H_\Lambda(\sigma_\Lambda\bar{\sigma}_{\Lambda^c})\Big)\prod_{j\in\Lambda}\alpha(d\sigma_j)}{\int_{S^\Lambda}\exp\Big(-H_\Lambda(\tilde{\sigma}_\Lambda\bar{\sigma}_{\Lambda^c})\Big)\prod_{j\in\Lambda}\alpha(d\tilde{\sigma}_j)}, \tag{33}$$

We now introduce a double-layer system or joint system by coupling the initial system to a second system (with single-spin space S') through the sitewise joint measures $K(d\sigma_i, d\eta_i)$ on $S \times S'$. Denote by $\tilde{\gamma}$ the specification of our new double-layer system, i.e. for a fixed boundary condition $\bar{\sigma} \in \Omega = S^G$ and a finite volume $\Lambda \subset G$, $\tilde{\gamma}_\Lambda(\cdot|\bar{\sigma}) \in \tilde{\gamma}$ is given by

$$\begin{aligned} \tilde{\gamma}_\Lambda(d\sigma_\Lambda, d\eta_\Lambda|\bar{\sigma}) &= \frac{\exp\Big(-H_\Lambda(\sigma_\Lambda\bar{\sigma}_{\Lambda^c})\Big)\prod_{j\in\Lambda}K(d\sigma_j, d\eta_j)}{\int_{(S\times S')^\Lambda}\exp\Big(-H_\Lambda(\tilde{\sigma}_\Lambda\bar{\sigma}_{\Lambda^c})\Big)\prod_{j\in\Lambda}K(d\tilde{\sigma}_j, d\tilde{\eta}_j)} \qquad (34) \\ &= \gamma_\Lambda(d\sigma_\Lambda|\bar{\sigma})\prod_{j\in\Lambda}K(d\eta_j|\sigma_j), \end{aligned}$$

where $K(d\eta_i|\sigma_i)$ denotes the K conditional distribution of the second spin given the value of the first. This specification is in general not Gibbs but in our case where we only have sitewise dependence between the two layers it is known for instance from [8] and references therein that $\tilde{\gamma}$ is Gibbs.

For each non-empty subset Λ of G we denote by $\mathcal{S}_\Lambda$ the collection of all non-empty finite subsets of Λ. We will write $\mathcal{S}$ instead of $\mathcal{S}_G$. For any fixed configuration $\sigma \in \Omega = S^G$ and any $\Lambda \in \mathcal{S}$ we define the finite-volume *transformed* distribution $\gamma'_{\Lambda,\bar{\sigma}}$ as

$$\gamma'_{\Lambda;\bar{\sigma}}(d\eta_\Lambda) := \int_{S^\Lambda}\tilde{\gamma}(d\sigma_\Lambda, d\eta_\Lambda|\bar{\sigma}_{\Lambda^c}). \tag{35}$$

It is important to note that in the joint system considered above, conditionally on the σ's the η's are independent. But taking the σ-average of the joint system creates dependence among the η's. Due to this dependence we now introduce finite-volume η conditional distributions by freezing the η configuration in the definition of $\gamma'_{\Lambda;\bar{\sigma}}$ except at some region $\Delta \in \mathcal{S}_\Lambda$. That is for any $\Lambda \in \mathcal{S}$ with $|\Lambda| \geq 2$ and $\Delta \in \mathcal{S}_\Lambda$ we have

$$\gamma'_{\Delta,\Lambda;\bar{\sigma}}(d\eta_\Delta|\bar{\eta}_{\Lambda\setminus\Delta}) = \frac{\int_{S^\Lambda}\exp\Big(-H_\Lambda(\sigma_\Lambda\bar{\sigma}_{\Lambda^c})\Big)\prod_{j\in\Lambda\setminus\Delta}K(d\sigma_j|\bar{\eta}_j)\prod_{i\in\Delta}K(d\sigma_i, d\eta_i)}{\int_{S^\Lambda}\exp\Big(-H_\Lambda(\sigma_\Lambda\bar{\sigma}_{\Lambda^c})\Big)\prod_{j\in\Lambda\setminus\Delta}K(d\sigma_j|\bar{\eta}_j)\prod_{i\in\Delta}\alpha(d\sigma_i)}. \tag{36}$$

The natural question that comes to mind is whether $\lim_{\Lambda\uparrow G}\gamma'_{\Delta,\Lambda;\bar{\sigma}}(d\eta_\Delta|\bar{\eta}_{\Lambda\setminus\Delta})$ exists for any fixed $\Delta \in \mathcal{S}_\Lambda$, $\bar{\sigma} \in \Omega$ and $\bar{\eta}_{\Delta^c} \in (S')^{G\setminus\Delta}$? If this limit exists we will denote

it by $\gamma'_\Delta(d\eta_\Delta|\bar{\eta}_{\Delta^c})$ and γ' by the class of all the conditional distributions for finite Δ. For the sake of simplicity we will always restrict our analysis to the case where Δ is a singleton. The analysis for general (but finite) Δ can be implemented using the same arguments used in the singleton case. It is our aim to provide a sufficient condition for the conditional probabilities $\gamma'_{i,\Lambda;\bar{\sigma}}(d\eta_i|\eta_{\Lambda\setminus i})$ to have an infinite-volume limit. For this we introduce the decomposition of the Hamiltonian H_Λ in the finite window Λ into its contributions coming from the sites in $\Lambda\setminus i$ and site i for any $i\in\Lambda$ as follows;

$$\begin{aligned} H_\Lambda(\sigma_\Lambda\bar{\sigma}_{\Lambda^c}) &= H_i(\sigma_\Lambda\bar{\sigma}_{\Lambda^c}) + H_{\Lambda\setminus i}(\sigma_{\Lambda\setminus i}\bar{\sigma}_{\Lambda^c}), \quad \text{where} \\ H_i(\sigma_\Lambda\bar{\sigma}_{\Lambda^c}) &= \sum_{A\ni i}\Phi_A(\sigma_\Lambda\bar{\sigma}_{\Lambda^c}) \quad \text{and} \\ H_{\Lambda\setminus i}(\sigma_{\Lambda\setminus i}\bar{\sigma}_{\Lambda^c}) &= \sum_{A\cap\Lambda\setminus i\neq\emptyset;\ i\notin A}\Phi_A(\sigma_{\Lambda\setminus i}\bar{\sigma}_{\Lambda^c}) \end{aligned} \tag{37}$$

We clearly see from the definition of an interaction that the Hamiltonian $H_{\Lambda\setminus i}$ is a function on the configuration space $S^{G\setminus i}$. For the infinite-volume transformed conditional distributions $\gamma'_i(d\eta_i|\eta_{i^c})$ to exist, some intermediate system living on the sublattice $G\setminus i$ must admit a unique infinite-volume Gibbs measure. This intermediate model is what we referred to as the ***restricted constrained first layer model*** (defined below w.r.t $H_{\Lambda\setminus i}$).

Definition 4.1 *The* ***restricted constrained first layer model (RCFLM)*** *in any $\Lambda\in\mathcal{S}$ with $|\Lambda|\geq 2$ and $i\in\Lambda$ is defined as the measure,*

$$\mu^{\bar{\sigma}}_{\Lambda\setminus i}[\eta_{\Lambda\setminus i}](d\sigma_{\Lambda\setminus i}) = \frac{\exp\Big(-H_{\Lambda\setminus i}(\sigma_{\Lambda\setminus i}\bar{\sigma}_{\Lambda^c})\Big)\prod_{j\in\Lambda\setminus i}K(d\sigma_j|\eta_j)}{\int_{S^{\Lambda\setminus i}}\exp\Big(-H_{\Lambda\setminus i}(\tilde{\sigma}_{\Lambda\setminus i}\bar{\sigma}_{\Lambda^c})\Big)\prod_{j\in\Lambda\setminus i}K(d\tilde{\sigma}_j|\eta_j)}, \tag{38}$$

for some $\bar{\sigma}=S^G$ and $\eta_\Lambda\in(S')^\Lambda$.

It is **restricted** because we only consider the spins in the sublattice $G\setminus i$ and **constrained** since we have frozen the configuration in the second layer . The RCFLM (as we will see from the lemma below) will provide us with a sufficient condition for the existence of an infinite-volume limit $\gamma'_i(d\eta_i|\eta_{i^c})$ for the conditional probabilities $\gamma'_{i,\Lambda;\bar{\sigma}}(d\eta_i|\eta_{\Lambda\setminus i})$.

Lemma 4.2 *Let $\Lambda\in\mathcal{S}$ with $|\Lambda|\geq 2$, then for any $i\in\Lambda$ and any $\bar{\sigma}\in\Omega$ we have*

$$\gamma'_{i,\Lambda;\bar{\sigma}}(d\eta_i|\eta_{\Lambda\setminus i}) = \frac{\int_{S^{\Lambda\setminus i}}\mu^{\bar{\sigma}}_{\Lambda\setminus i}[\eta_{\Lambda\setminus i}](d\sigma_{\Lambda\setminus i})\int_S\exp\Big(-H_i(\sigma_\Lambda\bar{\sigma}_{\Lambda^c})\Big)K(d\sigma_i,d\eta_i)}{\int_{S^{\Lambda\setminus i}}\mu^{\bar{\sigma}}_{\Lambda\setminus i}[\eta_{\Lambda\setminus i}](d\sigma_{\Lambda\setminus i})\int_S\exp\Big(-H_i(\sigma_\Lambda\bar{\sigma}_{\Lambda^c})\Big)d\alpha(\sigma_i)}. \tag{39}$$

Proof: By using the decomposition of H_Λ in (37) we can write $\gamma'_{i,\Lambda;\bar{\sigma}}(d\eta_i|\eta_{\Lambda\setminus i})$ as;

$$\gamma'_{i,\Lambda;\bar{\sigma}}(d\eta_i|\eta_{\Lambda\setminus i}) = \frac{\int_{S^{\Lambda\setminus i}}\exp\Big(-H_{\Lambda\setminus i}(\sigma_{\Lambda\setminus i}\bar{\sigma}_{\Lambda^c})\Big)\prod_{j\in\Lambda\setminus i}K(d\sigma_j|\eta_j)\int_S\exp\Big(-H_i(\sigma_\Lambda\bar{\sigma}_{\Lambda^c})\Big)K(d\sigma_i,d\eta_i)}{\int_{S^{\Lambda\setminus i}}\exp\Big(-H_{\Lambda\setminus i}(\sigma_{\Lambda\setminus i}\bar{\sigma}_{\Lambda^c})\Big)\prod_{j\in\Lambda\setminus i}K(d\sigma_j|\eta_j)\int_{S\times S'}\exp\Big(-H_i(\sigma_\Lambda\bar{\sigma}_{\Lambda^c})\Big)K(d\sigma_i,d\tilde{\eta}_i)}. \tag{40}$$

The claim of the lemma follows by multiplying the expression for $\gamma'_{i,\Lambda;\bar{\sigma}}(d\eta_i|\eta_{\Lambda\setminus i})$ above by $\dfrac{\int_{S^{\Lambda\setminus i}} \exp\Big(-H_{\Lambda\setminus i}(\tilde{\sigma}_{\Lambda\setminus i}\bar{\sigma}_{\Lambda^c})\Big)\prod_{j\in\Lambda\setminus i} K(d\tilde{\sigma}_j|\eta_j)}{\int_{S^{\Lambda\setminus i}} \exp\Big(-H_{\Lambda\setminus i}(\tilde{\sigma}_{\Lambda}\bar{\sigma}_{\Lambda^c})\Big)\prod_{j\in\Lambda\setminus i} K(d\tilde{\sigma}_j|\eta_j)}$ and simplifying the resulting expression.

□

It is not hard to infer from the above lemma that there will be an infinite-volume kernel $\gamma'_i(d\eta_i|\eta_{i^c})$ if the RCFLM has a unique infinite-volume Gibbs measure $\mu_{i^c}[\eta_{i^c}]$. This is the case since H_i is a local function which is finite by assumption. This was also observed in the corresponding mean-field set-up in [18]. Over there a sufficient condition for the existence of infinite-volume transformed kernel was given in terms of the uniqueness of global minimizers for some potential function. This condition was shown to be equivalent to the differentiability of the transformed Hamiltonian. We now state a result concerning an upper bound for Dobrushin's constant for the RCFLM.

Proposition 4.3 *Let the Dobrushin's interdependence matrix for the RCFLM for some fixed site $i_o \in G$ be the matrix whose entries are given by*

$$C^{i_o}_{ij}[\eta_i] = \sup_{\zeta,\bar{\zeta}\in S^{G\setminus i_o};\, \zeta_{j^c}=\bar{\zeta}_{j^c}} \Big\|\mu^{\zeta}_i[\eta_i] - \mu^{\bar{\zeta}}_i[\eta_i]\Big\|, \tag{41}$$

for any pair $i,j \in G\setminus i_o$ where we have denoted $\mu^{\bar{\zeta}}_i$ by the single-site part of $\mu^{\bar{\zeta}}_{\Lambda\setminus i_o}$. Then we have;

$$C^{i_o}_{ij}[\eta_i] \le \exp\Big(\sum_{A\supset\{i,j\};\, i_o\notin A} \delta(\Phi_A)\Big)\, dev_{\alpha_{\eta_i};i,j}(H_i). \tag{42}$$

where $\alpha_{\eta_i}(d\sigma_i) = K(d\sigma_i|\eta_i)$.
Furthermore, defining the Dobrushin constant $c'[\eta]$ for the RCFLM as

$$\begin{aligned} c'[\eta] &:= \sup_{i_o\in G} c^{i_o}[\eta], \quad \textit{with} \\ c^{i_o}[\eta] &= \sup_{i\in G\setminus i_o}\sum_{j\in G\setminus i_o} C^{i_o}_{ij}[\eta_i], \end{aligned} \tag{43}$$

we also have

$$\begin{aligned} c'[\eta] &\le \sup_{i_o\in G}\sup_{i\in G\setminus i_o}\sum_{j\in G\setminus i_o}\exp\Big(\sum_{A\supset\{i,j\};\, i_o\notin A}\delta(\Phi_A)\Big)\, dev_{\alpha_{\eta_i};i,j}(H_i) \\ &\le \sup_{i\in G}\sum_{j\in G}\exp\Big(\sum_{A\supset\{i,j\}}\delta(\Phi_A)\Big)\, dev_{\alpha_{\eta_i};i,j}(H_i)). \end{aligned} \tag{44}$$

In the case of $G = \mathbb{Z}^d$ and translation-invariant initial interactions the last inequality is an equality.

Proof: The proof follows the same lines as the proof of Theorem 2.2 but here we use $\alpha_{\eta_i} = K(\cdot|\eta_i)$ instead of α .

$\square$

It is also not hard to deduce from Proposition 3.2 that;

$$c'[\eta] \le \frac{1}{2} \sup_{i_o\in G} \sup_{i\in G\setminus i_o} \sum_{j\in G\setminus i_o} \exp\Big(\frac{1}{2} \sum_{A\supset\{i,j\};\, i_o\notin A} \delta(\Phi_A)\Big) \mathrm{std}_{\alpha_{\eta_i};i,j}(H_i) \qquad (45)$$
$$\le \frac{1}{2} \sup_{i\in G} \sum_{j\in G} \exp\Big(\frac{1}{2} \sum_{A\supset\{i,j\}} \delta(\Phi_A)\Big) \mathrm{std}_{\alpha_{\eta_i};i,j}(H_i).$$

Again Lipschitzness of the initial Hamiltonian carries over nicely.

Corollary 4.4 *Suppose the Lipschitz-condition (24). Then we have*

$$c'[\eta] = \sup_{i_o\in G} \sup_{i\in G\setminus i_o} \sum_{j\in G\setminus i_o} C_{ij}^{i_o}[\eta_i]$$
$$\le \frac{1}{2} \sup_{i_o\in G} \sup_{i\in G\setminus i_o} \sum_{j\in G\setminus i_o} \exp\Big(\frac{1}{2} \sum_{A\supset\{i,j\};\, i_o\notin A} \delta(\Phi_A)\Big) L_{ij} \inf_{a_i\in S} \Big(\int_S d^2(\sigma_i, a_i)\alpha_{\eta_i}(d\sigma_i)\Big)^{\frac{1}{2}} \qquad (46)$$
$$\le \frac{1}{2} \sup_{i\in G} \sum_{j\in G} \exp\Big(\frac{1}{2} \sum_{A\supset\{i,j\}} \delta(\Phi_A)\Big) L_{ij} \inf_{a_i\in S} \Big(\int_S d^2(\sigma_i, a_i)\alpha_{\eta_i}(d\sigma_i)\Big)^{\frac{1}{2}}.$$

The claim of the corollary follows from Corollary 3.3.

We now proceed to prove Theorem 2.6, but before we do this we still need some results from which the proof will follow. As a first step we recall some known results about Dobrushin's uniqueness concerning an estimate of the distance between the unique Gibbs measure admitted by a Gibbs specification satisfying Dobrushin's condition and another Gibbs measure corresponding to some other specification. This estimate tells us the local variation between the two infinite-volume probability measures. This result which we state in the proposition below can be found for example in [1] as Theorem 8.20. Before we state the result we fix some notations. Suppose $C(\gamma)$ is the Dobrushin interdependence matrix of a specification γ and $C^n(\gamma)$, $n \ge 0$, the n'th power of $C(\gamma)$, then we define the matrix

$$D(\gamma) = (D_{ij})_{i,j\in G} := \sum_{n\ge 0} C^n(\gamma). \qquad (47)$$

Proposition 4.5 *Let γ and $\bar\gamma$ be any two specifications with γ satisfying Dobrushin's condition. Suppose that for each $i \in G$ we have a measurable function b_i on the standard Borel space Ω with the property that*

$$||\gamma_i(\cdot|\sigma_{i^c}) - \bar\gamma_i(\cdot|\sigma_{i^c})|| \le b_i(\sigma) \qquad (48)$$

for all $\sigma \in \Omega$. Then for $\mu \in \mathcal{G}(\gamma)$ and $\bar\mu \in \mathcal{G}(\bar\gamma)$ we have

$$|\mu(f) - \bar\mu(f)| \le \sum_{i,j\in G} \delta_i(f) D_{ij}(\gamma)\bar\mu(b_j) \qquad (49)$$

for all functions f which are the uniform limits of functions that depend on finitely many local variables σ_i.

Observe from Lemma 4.2 that if the $RCFLM$ satisfies Dobrushin's condition uniformly in η the infinite-volume single-site kernels $\gamma_i'(\cdot|\eta_i^c)$ exist for every η . We will adapt the result in Proposition 4.5 to our present set-up to compare $\gamma_i'(\cdot|\eta_{i^c})$ and $\gamma_i'(\cdot|\bar{\eta}_{i^c})$ for any pair of configurations $\eta, \bar{\eta} \in \Omega' = (S')^G$. Further we denote by $\gamma[\eta_{i^c}]$ the specification of the RCFLM with full η_{i^c} configuration. Again we assume for the first layer model that $\mu = \lim_n \mu_{\Lambda_n}^{\bar{\sigma}}$ as in the hypothesis of Theorem 2.6.

Proposition 4.6 *Suppose the RCFLM on the sublattice $G \setminus i$ (for some $i \in G$) satisfies Dobrushin's condition uniformly in η with unique infinite-volume limit $\mu_{i^c}[\eta_{i^c}]$. Then*

1. *the second layer system (the transformed model) has infinite-volume single-site conditional distributions $\gamma_i'(d\eta_i|\eta_{i^c})$ given by*

$$\gamma_i'(d\eta_i|\eta_{i^c}) = \frac{\int_{S^{G\setminus i}} \mu_{i^c}[\eta_{i^c}](d\sigma_{i^c}) \int_S \exp\Big(-H_i(\sigma_i\sigma_{i^c})\Big) K(d\sigma_i, d\eta_i)}{\int_{S^{G\setminus i}} \mu_{i^c}[\eta_{i^c}](d\sigma_{i^c}) \int_S \exp\Big(-H_i(\sigma_i\sigma_{i^c})\Big) \alpha(d\sigma_i)} \tag{50}$$

2. *for any pair $\eta_{i^c}, \bar{\eta}_{i^c} \in (S')^{G\setminus i}$ we have for any $j \neq i$ that*

$$\Big\|\gamma_j[\eta_j](\cdot|\bar{\sigma}_{G\setminus i}) - \gamma_j[\bar{\eta}_j](\cdot|\bar{\sigma}_{G\setminus i})\Big\| \leq 2\exp\Big(\sum_{A\ni j} \delta_j(\Phi_A)\Big) \Big\|K(\cdot|\eta_j) - K(\cdot|\bar{\eta}_j)\Big\|, \tag{51}$$

   where the $\gamma_j[\eta_j](\cdot|\bar{\sigma}_{G\setminus i,})$'s are the single-site parts of the specification for the RCFLM for $i \in G$ and η_{i^c}, and

3. *given $h_2(\sigma_{i^c}) = \int_{S\times S'} K(d\sigma_i, d\eta_i) \exp\Big(H_i(\sigma_i\sigma_{i^c})\Big)$ it follows that*

$$\begin{aligned} &\Big|\mu_{i^c}[\eta_{i^c}](h_2) - \mu_{i^c}[\bar{\eta}_{i^c}](h_2)\Big| \leq 2e^{\sum_{A\ni i}\|\Phi_A\|_\infty} \\ &\times \sum_{k,j\in G\setminus i} \delta_k\Big(\sum_{A\supset\{i,k\}} \Phi_A\Big) \bar{D}_{kj} \exp\Big(\sum_{A\ni j} \delta_j(\Phi_A)\Big) \Big\|K(\cdot|\eta_j) - K(\cdot|\bar{\eta}_j)\Big\|. \end{aligned} \tag{52}$$

4. *Furthermore, for any $k \neq i$ it is the case that*

$$\delta_k\Big(h_2(\sigma_{i^c})\Big) \leq \delta_k\Big(\sum_{A\ni i,k} \Phi_A\Big) e^{\sum_{A\ni i}\|\Phi_A\|_\infty} \tag{53}$$

5. *and finally*

$$\Big\|\gamma_i'(d\eta_i|\eta_{i^c}) - \gamma_i'(d\eta_i|\bar{\eta}_{i^c})\Big\| \leq 2\frac{\Big|\mu_{i^c}[\eta_{i^c}](h_2) - \mu_{i^c}[\bar{\eta}_{i^c}](h_2)\Big|}{\mu_{i^c}[\bar{\eta}_{i^c}](h_2)} \tag{54}$$

Remark: In particular, we can write for any finite volume the corresponding relation for the finite-volume conditional distribution with full η-conditioning as in (50), i.e. if $\Delta \in \mathcal{S}$ the we have

$$\gamma_\Delta'(d\eta_\Delta|\eta_{\Delta^c}) = \frac{\int_{S^{G\setminus\Delta}} \mu_{\Delta^c}[\eta_{\Delta^c}](d\sigma_{\Delta^c}) \int_{S^\Delta} \exp\Big(-H_\Delta(\sigma_\Delta\sigma_{\Delta^c})\Big) \prod_{i\in\Delta} K(d\sigma_i, d\eta_i)}{\int_{S^{G\setminus\Delta}} \mu_{\Delta^c}[\eta_{\Delta^c}](d\sigma_{\Delta^c}) \int_{S^\Delta} \exp\Big(-H_\Delta(\sigma_\Delta\sigma_{\Delta^c})\Big) \prod_{i\in\Delta} \alpha(d\sigma_i)}. \tag{55}$$

Proof:

1. The proof follows from a two-step limiting procedure. We fix an η-conditioning only in a finite volume Δ and construct the infinite-volume measure of the RCFLM by fixing a boundary condition on the first layer outside Λ (which we assume for simplicity to contain Δ) and let Λ tend to infinity. Then we let Δ tend to infinity, and recover the conditional probabilities by Martingale convergence and uniform approximation of the infinite-volume RCFLM, with conditionings only in volume Δ.

   More precisely, it follows as in Lemma 4.2 that we have for finite-volume conditionings the representation

$$\gamma'_{i,\Delta,\Lambda,\bar{\sigma}}(d\eta_i|\eta_{\Delta\setminus i}) = \frac{\mu^{\bar{\sigma}}_{\Lambda\setminus i}[\eta_{\Delta\setminus i}]\Big[\int_S e^{-H_i(\sigma_i\bar{\sigma}_{\Lambda^c}\cdot_{\Lambda\setminus i})}K(d\sigma_i, d\eta_i)\Big]}{\mu^{\bar{\sigma}}_{\Lambda\setminus i}[\eta_{\Delta\setminus i}]\Big[\int_S e^{-H_i(\sigma_i\bar{\sigma}_{\Lambda^c}\cdot_{\Lambda\setminus i})}\alpha(d\sigma_i)\Big]} \tag{56}$$

   On the r.h.s. we see a RCFLM $\mu^{\bar{\sigma}}_{\Lambda\setminus i}[\eta_{\Delta\setminus i}]$ appearing with constrained measure α_{η_i} only in the volume $\Delta\setminus i$, i.e.

$$\mu^{\bar{\sigma}}_{\Lambda\setminus i}[\eta_{\Delta\setminus i}](d\sigma_\Lambda) = \frac{e^{-H_{\Lambda\setminus i}(\sigma_{\Lambda\setminus i}\bar{\sigma}_{\Lambda^c})}\prod_{i\in\Delta\setminus i}K(d\sigma_i|\eta_i)\prod_{i\in\Lambda\setminus\Delta}\alpha(d\sigma_i)}{\int_{S^{\Lambda\setminus i}} e^{-H_{\Lambda\setminus i}(\tilde{\sigma}_{\Lambda\setminus i}\bar{\sigma}_{\Lambda^c})}\prod_{i\in\Delta\setminus i}K(d\tilde{\sigma}_i|\eta_i)\prod_{i\in\Lambda\setminus\Delta}\alpha(d\tilde{\sigma}_i)} \tag{57}$$

   By the assumption of Theorem 2.6 we can construct the measures on the first layer as an infinite-volume limit with boundary condition $\bar{\sigma}$.

   Hence, the conditional distribution $\gamma'_{i,\Lambda,\bar{\sigma}}(d\eta_i|\eta_{\Delta\setminus i})$ has an infinite-volume limit $\gamma'_{i,\bar{\sigma}}(d\eta_i|\eta_{\Delta\setminus i})$, for any arbitrary conditioning $\eta_{\Delta\setminus i}$, since $h(\sigma_{\Lambda\setminus i}\bar{\sigma}_{\Lambda^c}, \eta_i) := \int_S e^{-H_i(\sigma_\Lambda\bar{\sigma}_{\Lambda^c})}k(\sigma_i,\eta_i)\alpha(d\sigma_i)$ is a bounded quasilocal function in σ for each η_i. Note that this conditional distribution still depends on the boundary condition $\bar{\sigma}$ when the initial specification is in the phase transition regime. Let us denote the corresponding specification of the RCFLM with η-conditioning only in $\Delta\setminus i$ by $\gamma[\eta_{\Delta\setminus i}]$. It follows from (54) that

$$\begin{aligned}&\Big\|\gamma'_{i,\bar{\sigma}}(d\eta_i|\eta_{\Delta\setminus i}) - \gamma'_i(d\eta_i|\eta_{i^c})\Big\| \\ &\le 2\frac{\Big|\mu_{i^c}[\eta_{\Delta\setminus i}]\Big[\int_{S'} h(\cdot,\eta_i)\alpha'(d\eta_i)\Big] - \mu_{i^c}[\eta_{i^c}]\Big[\int_{S'} h(\cdot,\eta_i)\alpha'(d\eta_i)\Big]\Big|}{\mu_{i^c}[\eta_{i^c}]\Big[\int_{S'} h(\cdot,\eta_i)\alpha'(d\eta_i)\Big]}.\end{aligned} \tag{58}$$

   But using the fact that

$$\|\gamma_j[\eta_{\Delta\setminus i}] - \gamma_j[\eta_{i^c}]\|\begin{cases} = 0 & \text{if } j\in\Delta\setminus i; \\ \le 2 & \text{if } j\in\Delta^c\end{cases} \tag{59}$$

   we have by the comparison criterion in Proposition 4.5 and using the assumption that the RCFLM with full η-conditioning satisfies Dobrushin's condition uniformly in η that

$$\begin{aligned}&\Big|\mu_{i^c}[\eta_{\Delta\setminus i}]\Big[\int_{S'} h(\cdot,\eta_i)\alpha'(d\eta_i)\Big] - \mu_{i^c}[\eta_{i^c}]\Big[\int_{S'} h(\cdot,\eta_i)\alpha'(d\eta_i)\Big]\Big| \\ &\le 2\sum_{i\in G}\sum_{j\in\Delta^c}\delta_i\Big(\int_{S'} h(\cdot,\eta_i)\alpha'(d\eta_i)\Big)\bar{D}_{ij}.\end{aligned} \tag{60}$$

Taking now the limit $\Delta \uparrow G$ we get (50), by weak convergence of the RCFLM in Δ to the full one, and by the backwards martingale convergence theorem. The convergence is weak since we require the single spin space to be separable and metrizable. In this set-up weak quasilocal topology is equivalent to weak topology.

2. The proof of assertion 2 utilizes the definition of the single-site part of the RCFLM and arbitrary test function g, with $|g| \leq 1$ to define

$$\left| \int g(\sigma_j)\Big(\gamma_j[\eta_j](d\sigma_j|\bar{\sigma}_{G\setminus i}) - \mu_j[\bar{\eta}_j](d\sigma_j|\bar{\sigma}_{G\setminus i})\Big)\right|. \tag{61}$$

The rest of the proof follows by adding and subtracting the following quantity

$$\frac{\int g(\sigma_j)\exp\Big(-H_j(\sigma_j\bar{\sigma}_{G\setminus\{i,j\}})\Big)K(d\sigma_j|\eta_j)\int \exp\Big(-H_j(\tilde{\sigma}_j\bar{\sigma}_{G\setminus\{i,j\}})\Big)K(d\tilde{\sigma}_j|\eta_j)}{\int \exp\Big(-H_j(\tilde{\sigma}_j\bar{\sigma}_{G\setminus\{i,j\}})\Big)K(d\tilde{\sigma}_j|\bar{\eta}_j)\int \exp\Big(-H_j(\tilde{\sigma}_j\bar{\sigma}_{G\setminus\{i,j\}})\Big)K(d\tilde{\sigma}_j|\eta_j)} \tag{62}$$

to the expression under the absolute value sign in (61), rearranging terms and simplifying appropriately.

3. It follows from (48) and (49) of Proposition 4.5 that

$$\begin{aligned}&\Big|\mu_{i^c}[\eta_{i^c}](h_2) - \mu_{i^c}[\bar{\eta}_{i^c}](h_2)\Big| \\ &\leq 2\sum_{k,j\in G\setminus i}\delta_k(h_2)\bar{D}_{kj}\exp\Big(\sum_{A\ni j}\delta_j(\Phi_A)\Big)\Big\|K(\cdot|\eta_j) - K(\cdot|\bar{\eta}_j)\Big\|,\end{aligned} \tag{63}$$

since by definition of H_i, h_2 is a local function on $S^{G\setminus i}$. The rest of the proof of 3 follows from the bound on $\delta_k(h_2)$ given in statement 4 of the Proposition.

4. Recalling that $h_2(\sigma_{i^c}) = \int_{S\times S'} K(d\sigma_i, d\eta_i)\exp\Big(-H_i(\sigma_i\sigma_{i^c})\Big)$ we estimate for any pair of configurations σ and $\bar{\sigma}$ that coincide except on k

$$\begin{aligned}&\Big|\exp\Big(-H_i(\sigma_i\sigma_{i^c})\Big) - \exp\Big(-H_i(\sigma_i\bar{\sigma}_{i^c})\Big)\Big| \\ &= \Big|\exp\Big(-\sum_{A\ni i,k}\Phi_A(\sigma_i\sigma_{i^c})\Big) - \exp\Big(-\sum_{A\ni i,k}\Phi_A(\sigma_i\bar{\sigma}_{i^c})\Big)\Big|\exp\Big(-\sum_{A\ni i,A\not\ni k}\Phi_A(\sigma_i\sigma_{i^c})\Big) \\ &\leq \delta_k\Big(\sum_{A\ni i,k}\Phi_A\Big)e^{\sum_{A\ni i}\|\Phi_A\|_\infty},\end{aligned} \tag{64}$$

where we have used the fact that $|e^x - e^y| \leq |x-y|e^{\max\{x,y\}}$.

5. Take a test function $\varphi : S' \to \mathbb{R}$, with $|\varphi| \leq 1$ and consider

$$\begin{aligned}&\int_{S'}\varphi(\eta_i)\Big(\gamma_i'(d\eta_i|\eta_{i^c}) - \gamma_i'(d\eta_i|\bar{\eta}_{i^c})\Big) \\ &= \frac{\int_{S^{G\setminus i}}\mu_{i^c}[\eta_{i^c}](d\sigma_{i^c})h_1(\sigma_{i^c})}{\int_{S^{G\setminus i}}\mu_{i^c}[\eta_{i^c}](d\sigma_{i^c})h_2(\sigma_{i^c})} - \frac{\int_{S^{G\setminus i}}\mu_{i^c}[\bar{\eta}_{i^c}](d\sigma_{i^c})h_1(\sigma_{i^c})}{\int_{S^{G\setminus i}}\mu_{i^c}[\bar{\eta}_{i^c}](d\sigma_{i^c})h_2(\sigma_{i^c})},\end{aligned} \tag{65}$$

where we have set $h_1(\sigma_{i^c}) = \int_{S\times S'} K(d\sigma_i, d\eta_i)\varphi(\eta_i)\exp\Big(-H_i(\sigma_i\sigma_{i^c})\Big)$. By adding and subtracting
$\frac{\mu_{i^c}[\eta_{i^c}](h_1)\mu_{i^c}[\eta_{i^c}](h_2)}{\mu_{i^c}[\eta_{i^c}](h_2)\mu_{i^c}[\bar{\eta}_{i^c}](h_2)}$ to the right hand side of (65) and making use of the fact that $||\varphi||_\infty \leq 1$ yields

$$\Big|\int_{S'}\varphi(\eta_i)\Big(\gamma_i'(d\eta_i|\eta_{i^c}) - \gamma_i'(d\eta_i|\bar{\eta}_{i^c})\Big)\Big| \leq 2\frac{\Big|\mu_{i^c}[\eta_{i^c}](h_2) - \mu_{i^c}[\bar{\eta}_{i^c}](h_2)\Big|}{\mu_{i^c}[\bar{\eta}_{i^c}](h_2)}. \tag{66}$$

□

Note from the proof of statement 5 of the above Proposition that the denominator in (66) can as well be $\mu_{i^c}[\eta_{i^c}](h_2)$ if one adds and subtracts from the right hand side of (65) $\frac{\mu_{i^c}[\bar{\eta}_{i^c}](h_1)\mu_{i^c}[\bar{\eta}_{i^c}](h_2)}{\mu_{i^c}[\eta_{i^c}](h_2)\mu_{i^c}[\bar{\eta}_{i^c}](h_2)}$ instead of $\frac{\mu_{i^c}[\eta_{i^c}](h_1)\mu_{i^c}[\eta_{i^c}](h_2)}{\mu_{i^c}[\eta_{i^c}](h_2)\mu_{i^c}[\bar{\eta}_{i^c}](h_2)}$, as was the case in the above proof. But any of the two makes no difference since in our estimate we don't make use of the actual integral of h_2 but instead we utilize its uniform norm. Having disposed of the results above, we now return to the

Proof of Theorem 2.6:

1. The proof follows from Lemma 4.2 and the unicity of the Gibbs measures admitted by the RCFLM, which is uniform in η.

2. Using (54) and (52) of Proposition 4.6 we get

$$\begin{aligned}\Big\|\gamma_i'(d\eta_i|\eta_{i^c}) - \gamma_i'(d\eta_i|\bar{\eta}_{i^c})\Big\| &\leq 2\frac{\Big|\mu_{i^c}[\eta_{i^c}](h_2) - \mu_{i^c}[\bar{\eta}_{i^c}](h_2)\Big|}{\mu_{i^c}[\bar{\eta}_{i^c}](h_2)} \\ &\leq 4e^{2\sum_{A\ni i}\|\Phi_A\|_\infty}\sum_{k,j\in G\setminus i}\delta_k\Big(\sum_{A\supset\{i,k\}}\Phi_A\Big)\bar{D}_{kj}e^{\sum_{A\ni j}\delta_j(\Phi_A)}\Big\|K(\cdot|\eta_j) - K(\cdot|\bar{\eta}_j)\Big\|.\end{aligned} \tag{67}$$

The 2 in front of $\sum_{A\ni i}\|\Phi_A\|_\infty$ in the exponential is obtained by observing that $\frac{1}{\mu_{i^c}[\bar{\eta}_{i^c}](h_2)} \leq \frac{1}{e^{-\sum_{A\ni i}\|\Phi_A\|_\infty}}$.

□

5 Proof of results on short-time Gibbsianness for time-evolved rotator models

Proof of Theorem 2.7: Consider the rotator model on the lattice G, with $S = S^{q-1}$ (the sphere in q-dimensional Euclidean space, with $q \geq 2$) as the spin space and Hamiltonian given by $H(\sigma) = \sum_{i,j\in G; i\neq j} J_{ij}\sigma_i\cdot\sigma_j$. We consider the RCFLM for this Hamiltonian with K given by the diffusion or the heat kernel k_t on the sphere, i.e. $K(d\sigma_i, d\eta_i) = K_t(d\sigma_i, d\eta_i) = k_t(\sigma,\eta)\alpha_o(d\sigma)\alpha_o(d\eta)$, where α_o is the equidistribution on S^{q-1}. In this case we have $S = S' = S^{q-1}$ and $\alpha_o(d\sigma_i) = \int_{\eta_i} K_t(d\sigma_i, d\eta_i)$. For the given Hamiltonian, $H_i(\cdot\zeta_{i^c}) - H_i(\cdot\bar{\zeta}_{i^c})$ is Lipschitz continuous with Lipschitz constant

$L_{ij} = 2|J_{ij}|$. To obtain the desired bound on the Dobrushin interdependence matrix entries we employ the bound given by Corollary 4.4. In view of this, we need to evaluate the integrals $\int_S d^2(\sigma_i, a_i) K(d\sigma_i|\eta_i) = \int_{S^{q-1}} d^2(\sigma_i, a_i) k_t(\sigma_i, \eta_i) \alpha_o(d\sigma_i)$. To compute this integrals we choose $a_i = \eta_i$ and denote by Z_t^q the q-th coordinate of a diffusion on the sphere started at $Z_{t=0}^q = 1$ (the "north-pole") and denote the corresponding expectation by $\mathbb{E}$. Thus for any η_i we have;

$$\int \alpha_o(d\sigma_i) k_t(\sigma_i, \eta_i) d^2(\sigma_i, \eta_i) = 2(1 - \mathbb{E} Z_t^q) = 2(1 - e^{-(q-1)t}). \tag{68}$$

The first equality uses the idea that Brownian motion on the sphere is rotation invariant and consequently choosing $\eta_i = (0, \cdots, 0, 1)$. To see the last equality use either an explicit form of the transition kernel k_t in polar coordinates and orthogonality of Legendre polynomials as in [2]. Or use that the generator of the diffusion Z_t^q given by the u-dependent parts of the Laplace-Beltrami operator on the sphere reads $(1-u^2)\left(\frac{d}{du}\right)^2 - (q-1)u\frac{d}{du}$ and generates the equation $\frac{d}{dt}\mathbb{E}Z_t^q = -(q-1)\mathbb{E}Z_t^q$. Solving with the initial condition $Z_{t=0}^q = 1$ yields the desired result. Note in our present set-up that for any pair $i, j \in G \setminus i_o$ we have $\sum_{A \supset \{i,j\}, i_o \notin A} \delta(\Phi_A) = 2|J_{ij}|$. Then it follows from Corollary 4.4 that

$$\begin{aligned} c'[\eta] &\leq \sqrt{2} \sup_{i_o \in G} \sup_{i \in G \setminus i_o} \sum_{j \in G \setminus i_o} \exp\big(|J_{ij}|\big)|J_{ij}| \left(1 - e^{-(q-1)t}\right)^{\frac{1}{2}} \\ &\leq \sqrt{2} \sup_{i \in G} \sum_{j \in G} e^{|J_{ij}|} |J_{ij}| \left(1 - e^{-(q-1)t}\right)^{\frac{1}{2}}. \end{aligned} \tag{69}$$

The above estimate on $c'[\eta]$ is uniform in η.

1. Therefore the proof of the Gibbsianness of the time-evolved measure μ_t follows from the above uniform estimate on $c'[\eta]$ and the hypothesis of the theorem.

2. An application of the continuity estimate on γ_i' in Theorem 2.6 to the rotator model yields a continuity estimate on $\gamma_{i,t}'$ when we define $Q_{ij}(t)$ by the bound on $Q_{i,j}$ in (11). Since the introduction of the Euclidean metric d follows from the estimate on the posterior metric d' found in Proposition 2.8 and the quantity $\bar{D}_{kj}$ appearing in the definition of the Q_{ij} in Theorem 2.6 is given by $\bar{D}_{kj}(t) = \left(\mathbf{1} + \sum_{n=1}^{\infty} \left(1 - e^{-(q-1)t}\right)^{\frac{n}{2}} A^n\right)_{kj}$ where $A_{ij} = e^{|J_{ij}|}|J_{ij}|$. It is also elementary to see that $\sum_{A \ni j} \delta_j(\Phi_A) \leq 2 \sum_{A \ni j} ||\Phi_A||_\infty$ and for each $i \in G$, $\sum_{A \ni i} ||\Phi_A||_\infty = \sum_{j \in G} |J_{ij}|$. Thus, putting all the above together we get

$$\left\|\gamma_{i,t}'(\cdot|\eta_{i^c}) - \gamma_{i,t}'(\cdot|\bar{\eta}_{i^c})\right\| \leq \sqrt{\frac{\pi}{4t}} \sum_{j \in G \setminus i} Q_{ij}(t) d(\eta_j, \bar{\eta}_j). \tag{70}$$

The rest of the proof follows from a telescoping argument involving the sites in $G \setminus i$. The main result in this direction that we will employ in our proof is formulated in the lemma below.

Lemma 5.1 *For each non-empty finite subset $V_1 \subset G \setminus i$ we have the following estimate*

$$\begin{aligned} \left\|\gamma_{i,t}'(\cdot|\eta_{i^c}) - \gamma_{i,t}'(\cdot|\bar{\eta}_{i^c})\right\| &\leq \frac{1}{2} \sum_{j \in V_1} \min\left\{\sqrt{\frac{\pi}{t}} Q_{ij}(t), e^{4\sum_{k \in G}|J_{jk}|} - 1\right\} d(\eta_j, \bar{\eta}_j) \\ &+ \left\|\gamma_{i,t}'(\cdot|\eta_{V_1^c \setminus i}\bar{\eta}_{V_1}) - \gamma_{i,t}'(\cdot|\bar{\eta}_{i^c})\right\|. \end{aligned} \tag{71}$$

Note from the second term in the above bound that the conditionings coincides in the chosen finite volume V_1. We proceed by appling the Lemma 5.1 to obtain a similar bound for $\|\gamma'_{i,t}(\cdot|\eta_{V_1^c\setminus i}\bar{\eta}_{V_1}) - \gamma'_{i,t}(\cdot|\bar{\eta}_{i^c})\|$ this time for any non-empty finite subset $V_2 \subset G \setminus V_1 \cup \{i\}$. Thus we have

$$\begin{aligned}\|\gamma'_{i,t}(\cdot|\eta_{i^c}) - \gamma'_{i,t}(\cdot|\bar{\eta}_{i^c})\| \leq \frac{1}{2}\sum_{j\in V_1\cup V_2} \min\left\{\sqrt{\frac{\pi}{t}}Q_{ij}(t), e^{4\sum_{k\in G}|J_{jk}|} - 1\right\} d(\eta_j, \bar{\eta}_j) \\ + \|\gamma'_{i,t}(\cdot|\eta_{(V_1\cup V_2)^c\setminus i}\bar{\eta}_{V_1\cup V_2}) - \gamma'_{i,t}(\cdot|\bar{\eta}_{i^c})\|.\end{aligned} \tag{72}$$

Successive application of Lemma 5.1 along such sequence of pair-wise disjoint non-empty finite subsets V_n such that $\cup_n V_n = G \setminus i$ yields the desired result.

□

Proof of Lemma 5.1:
For any non-empty finite subset $\Lambda \subset G\setminus i$ we let $n_\Lambda : \Lambda \longrightarrow \{1, 2, \cdots, |\Lambda|\}$ be a bijection between Λ and $\{1, 2, \cdots, |\Lambda|\}$ and denote by $\bar{\eta}_{l\leq}\eta$ the configuration that coincides with $\bar{\eta}$ on $n_\Lambda^{-1}(\{1, \cdots, l\})$ and η on $G \setminus n_\Lambda^{-1}(\{1, \cdots, l\}) \cup \{i\}$. The map n_Λ orders the elements in Λ. For $G = Z^2$ this map can be a spiral ordering of the sites in Λ. Recall that the joint a priori measure $K_t(d\sigma_i, d\eta_i) = k_t(\sigma_i, \eta_i)\alpha_o(d\sigma_i)\alpha_o(d\eta_i)$ where as before $\alpha_o = \int K_t(\cdot, d\sigma_i)$. In this way we can write the single-site part of γ' as;

$$\begin{aligned}&\gamma'_{i,t}(d\eta_i|\eta_{i^c}) = f(\eta_i|\eta_{i^c})\alpha_o(d\eta_i), \quad \text{where} \\ &f(\eta_i|\eta_{i^c}) = \frac{\int_{S^{G\setminus i}} \mu_{i^c}[\eta_{i^c}](d\tilde{\sigma}_{i^c}) \int_S \exp\Big(-H_i(\sigma_i\tilde{\sigma}_{i^c})\Big) k_t(\sigma_i, \eta_i)\alpha_o(d\sigma_i)}{\int_{S^{G\setminus i}} \mu_{i^c}[\eta_{i^c}](d\tilde{\sigma}_{i^c}) \int_S \exp\Big(-H_i(\sigma_i\tilde{\sigma}_{i^c})\Big)\alpha_o(d\sigma_i)}.\end{aligned} \tag{73}$$

With the order on Λ we can now write for any pair of conditionings $\eta, \bar{\eta} \in \Omega' = (S')^G$

$$f(\eta_i|\eta_{i^c}) - f(\eta_i|\bar{\eta}_{i^c}) = \sum_{l=1}^{|\Lambda|+1} \nabla_l f(\eta_i|\eta_{i^c}, \bar{\eta}_{i^c}) \quad \text{with} \tag{74}$$

$$\tag{75}$$

$$\nabla_l f(\eta_i|\eta_{i^c}, \bar{\eta}_{i^c}) = \begin{cases} f(\eta_i|\bar{\eta}_{l-1\leq}\eta) - f(\eta_i|\bar{\eta}_{l\leq}\eta) & \text{if } 1 \leq l \leq |\Lambda|; \\ f(\eta_i|\bar{\eta}_{|\Lambda|\leq}\eta) - f(\eta_i|\bar{\eta}_{i^c}) & \text{if } l = |\Lambda| + 1, \end{cases}$$

where we assume $\{1, \cdots, l-1\} = \emptyset$ for $l = 1$. In this spirit it follows from the triangle inequality that

$$\begin{aligned}&\|\gamma'_{i,t}(\cdot|\eta_{i^c}) - \gamma'_{i,t}(\cdot|\bar{\eta}_{i^c})\| = \int \alpha_o(d\eta_i)\Big|\sum_{l=1}^{|\Lambda|+1} \nabla_l f(\eta_i|\eta_{i^c}, \bar{\eta}_{i^c})\Big| \\ &\leq \sum_{l=1}^{|\Lambda|} \int_{S'} \alpha_o(d\eta_i)\big|\nabla_l f(\eta_i|\eta_{i^c}, \bar{\eta}_{i^c})\big| + \|\gamma'_{i,t}(\cdot|\bar{\eta}_\Lambda\eta_{G\setminus\{i\}\cup\Lambda}) - \gamma'_{i,t}(\cdot|\eta_{i^c})\|.\end{aligned} \tag{76}$$

To get the desired bound for the first term in the above inequality we use two estimation procedures which provide bounds for the terms in the sum that are multiples of $d(\eta_j, \bar{\eta}_j)$.

As a first step we consider for any $1 \leq l \leq |\Lambda|$ an estimate similar to the one given in (9) but here we define $Q_{ij}(t)$ by the bound in (11). Note for $1 \leq l \leq |\Lambda|$ the

conditionings in the definition of $\nabla_l f(\cdot|\eta_{i^c},\bar{\eta}_{i^c})$ coincide except at the site $j = n_\Lambda^{-1}(l)$. Thus it follows from (9) and the estimate on the posterior metric in Proposition 2.8 that for each $1 \leq l \leq |\Lambda|$

$$\big\|\gamma'_{i,t}(\cdot|\bar{\eta}_{l-1\leq}\eta) - \gamma'_{i,t}(\cdot|\bar{\eta}_{l\leq}\eta)\big\| = \int_{S'} \alpha_o(d\eta_i)\big|\nabla_l f(\eta_i|\eta_{i^c},\bar{\eta}_{i^c})\big| \leq \sqrt{\frac{\pi}{4t}} Q_{ij}(t) d(\eta_j,\bar{\eta}_j). \tag{77}$$

Next we apply the following estimation technique to obtain a second bound on $\big\|\gamma'_{i,t}(\cdot|\bar{\eta}_{l-1\leq}\eta) - \gamma'_{i,t}(\cdot|\bar{\eta}_{l\leq}\eta)\big\|$ for $1 \leq l \leq |\Lambda|$. First we set $j = n_\Lambda^{-1}(l)$ and note that

$$\frac{f(\eta_i|\bar{\eta}_{l-1\leq}\eta)}{f(\eta_i|\bar{\eta}_{l\leq}\eta)} = \frac{f(\eta_j|\eta_i\bar{\eta}_{l-1\leq}\eta_{l>})}{f(\bar{\eta}_j|\eta_i\bar{\eta}_{l-1\leq}\eta_{l>})} \times \frac{\int_{S'} f(\eta_i,\bar{\eta}_j|\bar{\eta}_{l-1\leq}\eta_{l>})\alpha_o(d\eta_i)}{\int_{S'} f(\eta_i,\eta_j|\bar{\eta}_{l-1\leq}\eta_{l>})\alpha_o(d\eta_i)}, \tag{78}$$

where $\bar{\eta}_{l-1\leq}\eta_{l>}$ is the configuration that coincides with $\bar{\eta}$ on $n_\Lambda^{-1}\big(\{1,\cdots,l-1\}\big)$ and η on $G\backslash n_\Lambda^{-1}\big(\{1,\cdots,l-1\}\big)\cup\{i,j\}$ and $f(\eta_i,\eta_j|\bar{\eta}_{l-1\leq}\eta_{l>})$ is given by (73) if we appropriately replace i in (73) with $\{i,j\}$.
Therefore setting $h_2(\sigma_{j^c},\eta_j) = \int_S \exp(-H_j(\sigma_j\sigma_{j^c}))k_t(d\sigma_j,\eta_j)\alpha_o(d\sigma_i)$ we have

$$\frac{f(\eta_j|\eta_i\bar{\eta}_{l-1\leq}\eta_{l>})}{f(\bar{\eta}_j|\eta_i\bar{\eta}_{l-1leq}\eta_{l>})} = \frac{\mu_{j^c}[\eta_i\bar{\eta}_{l-1\leq}\eta_{l>}]\Big[h_2(\sigma_{j^c},\eta_j)\Big]}{\mu_{j^c}[\eta_i\bar{\eta}_{l-1\leq}\eta_{l>}]\Big[h_2(\sigma_{j^c},\bar{\eta}_j)\Big]}. \tag{79}$$

Let R be a rotation such that $R\bar{\eta}_j = \eta_j$ and set $\sigma'_j = R\sigma_j$. Then it follows from the fact that $|H_j(\sigma_j\sigma_{j^c}) - H_j(\sigma'_j\sigma_{j^c})| \leq \Big(\sum_{k\in G}|J_{jk}|\Big)d(\eta_j,\bar{\eta}_j)$

$$\begin{aligned} &h_2(\sigma_{j^c},\eta_j) = \\ &\int_S \Big\{\int_S \exp\Big(-\Big(H_j(\sigma_j\sigma_{j^c}) - H_j(\sigma'_j\sigma_{j^c})\Big) - H_j(\sigma'_j\sigma_{j^c})\Big)K_t(d\sigma'_j|\eta_j)\Big\}K_t(d\sigma_j|\eta_j) \\ &\leq \exp\Big(c_j\ d(\eta_j,\bar{\eta}_j)\Big)\int_S \exp\Big(-H_j(\sigma'_j\sigma_{j^c})\Big)K_t(d\sigma'_j|\eta_j) \quad \text{and} \quad \text{similarly} \\ &h_2(\sigma_{j^c},\bar{\eta}_j) \leq \exp\Big(c_j\ d(\eta_j,\bar{\eta}_j)\Big)\int_S \exp\Big(-H_j(\sigma_j\sigma_{j^c})\Big)K_t(d\sigma_j|\eta_j) \end{aligned} \tag{80}$$

where $c_j = \sum_{k\in G}|J_{jk}|$. It follows from (79) and the rotation invariance of K_t that

$$\begin{aligned} &\frac{f(\eta_j|\eta_i\bar{\eta}_{l-1\leq}\eta_{l>})}{f(\bar{\eta}_j|\eta_i\bar{\eta}_{l-1\leq}\eta_{l>})} \leq \frac{\mu_{j^c}[\eta_i\bar{\eta}_{l-1\leq}\eta_{l>}]\Big[\exp\Big(c_j\ d(\eta_j,\bar{\eta}_j)\Big)\int_S \exp\Big(-H_j(\sigma'_j\sigma_{j^c})\Big)K_t(d\sigma'_j|\eta_j)\Big]}{\mu_{j^c}[\eta_i\bar{\eta}_{l-1\leq}\eta_{l>}]\Big[\int_S \exp(-H_j(\sigma'_j\sigma_{j^c}))k_t(d\sigma'_j,\eta_j)\alpha_o(d\sigma_i)\Big]} \\ &= e^{c_j d(\eta_j,\bar{\eta}_j)}. \end{aligned} \tag{81}$$

The above estimate follows by applying the rotation R to the $\bar{\eta}_j$ in the r.h.s. of (79). Furthermore, it is not hard to deduce that

$$\frac{\int_{S'} f(\eta_i,\bar{\eta}_j|\bar{\eta}_{l-1\leq}\eta_{l>})\alpha_o(d\eta_i)}{\int_{S'} f(\eta_i,\eta_j|\bar{\eta}_{l-1\leq}\eta_{l>})\alpha_o(d\eta_i)} \leq \sup_{\eta_i}\frac{f(\bar{\eta}_j|\eta_i\bar{\eta}_{l-1\leq}\eta_{l>})}{f(\eta_j|\eta_i\bar{\eta}_{l-1\leq}\eta_{l>})} \leq e^{c_j d(\eta_j,\bar{\eta}_j)}. \tag{82}$$

Therefore it follows from (78) that

$$\frac{f(\eta_i|\bar{\eta}_{l-1\leq}\eta)}{f(\eta_i|\bar{\eta}_{l\leq}\eta)} \leq e^{2c_j d(\eta_j,\bar{\eta}_j)}. \tag{83}$$

Hence for any $1 \leq l \leq |\Lambda|$ we have

$$\begin{aligned} &\int_S \alpha_o(d\eta_i)\Big|\nabla_l f(\eta_i|\eta_{i^c},\bar\eta_{i^c})\Big| = \int_S \alpha_o(d\eta_i)\Big|\Big(\frac{f(\eta_i|\bar\eta_{l-1\leq}\eta)}{f(\eta_i|\bar\eta_{l\leq}\eta)}-1\Big)f(\eta_i|\bar\eta_{l\leq}\eta)\Big| \\ &\leq e^{2c_j d(\eta_j,\bar\eta_j)}-1 \leq \frac{e^{4c_j}-1}{2}d(\eta_j,\bar\eta_j). \end{aligned} \tag{84}$$

Comparing (77) and (84) it is clearly seen for any $1 \leq l \leq |\Lambda|$ with $j = n_\Lambda^{-1}(l)$ that

$$\big\|\gamma'_{i,t}(\cdot|\bar\eta_{l-1\leq}\eta)-\gamma'_{i,t}(\cdot|\bar\eta_{l\leq}\eta)\big\| \leq \frac{1}{2}\min\Big\{\sqrt{\frac{\pi}{t}}Q_{ij}(t), e^{4\sum_{k\in G}|J_{jk}|}-1\Big\}d(\eta_j,\bar\eta_j), \tag{85}$$

which proves the lemma. □

Lemma 5.1 has an extension for interactions for which $H_j(\cdot\sigma_{j^c})$ is not Lipschitz continuous. In this set-up we have for any non-empty finite subset $V \subset G\setminus i$

$$\big\|\gamma'_{i,t}(\cdot|\eta_{i^c})-\gamma'_{i,t}(\cdot|\bar\eta_{i^c})\big\| \leq \sum_{j\in V}\left(e^{4\delta_j\left(\sum_{A\ni j}\Phi_A\right)}-1\right)+\big\|\gamma'_{i,t}(\cdot|\eta_{V^c\setminus i}\bar\eta_V)-\gamma'_{i,t}(\cdot|\bar\eta_{i^c})\big\|. \tag{86}$$

To obtain the desired bound on the posterior metric we need to solve the diffusion equation on the sphere S^{q-1}. However, it turns out in the analysis that we don't need all the components of the diffusion to arrive at our desired bound. The only coordinate that we will be interested in, is the qth, i.e. we only have to solve the resulting diffusion equation for the qth component. We employ both analytical and stochastic differential equation (sde) techniques to arrive at the diffusion of interest. It turns out that the sde approach easily provides the desired bound. Nevertheless, we present the analytical approach because of its interest per se. We first state the corresponding sde result.

Lemma 5.2 *1. Denote by Z_t the qth-component of the diffusion on the sphere S^{q-1} for $q \geq 2$, started at a value $\sin\varphi_0$ with $\varphi_0 \in (0,\frac{\pi}{2})$. Then there is a coupling of Z_t to a Brownian motion on the line, B_t such that the first passage time of Z_t at zero, denoted by $T_0(Z)$ is dominated from above by that of $\varphi_0+\sqrt{2}B_t$.*

2. Consequenty, independently of the dimension $q-1$ there is the estimate

$$P(T_0(Z)\geq t) \leq P(T_0(\varphi_0+\sqrt{2}B.)\geq t) \leq 2P\Big(0\leq G\leq \frac{\varphi_0}{\sqrt{2t}}\Big) \tag{87}$$

where G is a standard normal variable.

Proof: Consider the case $q \geq 3$ first. The sde for the q-th component reads,

$$dZ_t = -(q-1)Z_t dt + \sqrt{2(1-Z_t^2)}dB_t \tag{88}$$

Consider the transformation

$$Z_t = \sin(\bar\varphi_t) \tag{89}$$

to an unknown function $\bar{\varphi}_t$ describing the elevation above the equator. We apply this transformation only for $0 < Z_t < 1$, and so there is a one-to-one map to $0 < \bar{\varphi}_t < \frac{\pi}{2}$. In this range the sde is equivalent to

$$d\bar{\varphi}_t = -(q-2)\tan\bar{\varphi}_t + \sqrt{2}\, dB_t \tag{90}$$

Indeed, for $q \geq 3$ the diffusion $\bar{\varphi}_t$ does not leave the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$, meaning that that, with probability one the northpole is never reached by Z_t. (That this is true can be seen by projecting Z_t along the q-th axis, onto the $q-1$-dimensional plane.)

Integrating from zero to t we obtain from (90)

$$\bar{\varphi}_t = -(q-2)\int_0^t \tan\bar{\varphi}_s ds + \sqrt{2}\, B_t + \varphi_0 \tag{91}$$

From this equality we see that as long as $\bar{\varphi}_s \geq 0$ for all $s \in [0,t]$ we have the bound $\bar{\varphi}_t \leq \sqrt{2}\, B_t + \varphi_0$. This shows that the first passage time of $\bar{\varphi}_t$ is not bigger than that of $\sqrt{2}\, B_t + \varphi_0$.

The proof of the inequality follows from bounding $P(T_0(Z) \geq t)$ from above by the first passage time of the Brownian motion on a line, $P(T_0(\sqrt{2}\, B. + \varphi_0) \geq t)$. The latter can be computed exactly by the reflection principle applied to standard Brownian motion, as it is well-known. (We will use the reflection principle also in the proof Lemma 5.4, applied to the diffusion on the sphere.) This gives rise to the estimate on the r.h.s.

That the inequality holds also in the case $q = 2$ (and is a strict inequality then) can be seen directly without making reference to the SDE. We note that the paths of a diffusion on the circle are given by Brownian motions on the angular variable, i.e. $\bar{\varphi}_t = \sqrt{2}\, B_t + \varphi_0$. Then $\bar{\varphi}_t = 0$ implies that $Z_t^{(2)} = \sin(\bar{\varphi}_t) = 0$, but the converse is not true.

It is interesting to realize that this construction provides a coupling such that $Z_t^{(q)} \leq \sqrt{2}\, B_t + \varphi_0$, for $q \geq 3$, $Z_t^{(2)} \leq \sqrt{2}\, B_t + \varphi_0$ but *not* $Z_t^{(q)} \leq Z_t^{(2)}$. The latter relation is guaranteed to hold only as long as $0 \leq \sqrt{2}\, B_t + \varphi_0 \leq \frac{\pi}{2}$. $\square$.

We now present an analytical treatment for the diffusions considered above. This involves the study of eigenvalue problem involving the qth-component of the Laplace-Beltrami operator on the sphere. In fact the resulting eigenfunctions solve the spatial part of the qth-component of the diffusion on the sphere. The transition kernel k_t(defined below) for the qth-component of the diffusion is determined by the solution for the above mentioned eigenvalue problem. It is known from the literature [2] that the Legendre polynomials constitute a complete class of eigenfunctions, i.e. the transition kernel k_t can be written in terms of the Legendre polynomials.

Definition 5.3 *The Legendre polynomial $P_n(q,\cdot)$ of degree n in dimension $q \geq 2$ is given by the Rodrigues formula*

$$P_n(q,s) := \frac{(-1)^n \Gamma\left(\frac{q-1}{2}\right)}{2^n \Gamma\left(n + \frac{q-1}{2}\right)} \left(1-s^2\right)^{\frac{3-q}{2}} \left(\frac{d}{ds}\right)^n \left(1-s^2\right)^{\frac{q-3}{2}+n}, \tag{92}$$

where $-1 \leq s \leq 1$.

These Legendre polynomials are known (see [2] for example) to be orthogonal and satisfy the second order differential equations

$$\Big[(1-s^2)\frac{d^2}{ds^2}-(q-1)s\frac{d}{ds}+n(n+q-2)\Big]P_n(q,s)=0. \tag{93}$$

The last equation indicates that the Legendre polynomials are eigenfunctions for the eigenvalue problem for the qth component of the Laplace-Beltrami operator on the sphere S^{q-1} . This implies that the transition kernel for the qth coordinate Z^q_t of the Brownian motion on S^{q-1} can be written as

$$\begin{aligned} k_t(s,u) &:= \frac{\Gamma\left(\frac{q}{2}\right)}{\sqrt{(\pi)}\Gamma\left(\frac{q-1}{2}\right)}\sum_{n=0}^{\infty} e^{-n(n+q-2)t}N(q,n)P_n(q,s)P_n(q,u), \quad \text{where} \\ N(q,n) &:= \begin{cases} \frac{(2n+q-2)\Gamma(n+q-2)}{\Gamma(n+1)\Gamma(q-1)} & \text{if } n\geq 1; \\ 1 & \text{if } n=0 \end{cases} \end{aligned} \tag{94}$$

is the dimension of spherical harmonics of degree n in dimension q. Further we have set $Z^q_0=s$ and $Z^q_t=u$, and we have also chosen the constant $\frac{\Gamma\left(\frac{q}{2}\right)}{\sqrt{(\pi)}\Gamma\left(\frac{q-1}{2}\right)}$ so that for any initial s the integral of $k_t(s,u)$ with respect to the invariant measure $(1-u^2)^{\frac{q-3}{2}}du$ (which is the q-coordinate projection of the invariant surface measure on the sphere) over the interval [-1,1] is equal to one. We now formulate our result on an estimate on the posterior metric $d'(\eta_j,\bar\eta_j)$ define in (2.5). This is given in terms of Legendre polynomials (introduced in Definition 5.3 above) which by our construction are also themselves functions of $d(\eta_j,\eta'_j)$(the Euclidean distance between η_j and η'_j).

Lemma 5.4 *For the diffusion on a sphere there is an estimate of the posterior-metric $d'(\eta,\eta')$ at fixed t in terms of $d(\eta,\eta')$, the induced metric on the sphere S^{q-1} obtained by imbedding the sphere into the Euclidean space, given by*

$$d'(\eta,\eta')\leq F_t(d(\eta_j,\bar\eta_j)) \tag{95}$$

with the function

$$\begin{aligned} F_t(x) &= 2\Big(1-2P^{\frac{x}{2}}(Z^q_t\leq 0)\Big) \\ &= \frac{-4\Gamma\left(\frac{q}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{q-1}{2}\right)}\sum_{n=1,3,5,\ldots} e^{-n(n+q-2)t}N(q,n)P_n\Big(q,\frac{x}{2}\Big)\int_{-1}^{0}P_n(q,s)(1-s^2)^{\frac{q-3}{2}}ds \end{aligned} \tag{96}$$

Proof:
The idea of the proof is to construct a coupling of two diffusions on the sphere starting at the points η and η' . By rotation invariance of such diffusions we assume that η and η' are mirror images of each other under reflection at the equatorial plane. Then we construct a coupling by reflection [13] of the path started at η with the equator as the mirror line, up to the time where the diffusion hits the equator. After that the two diffusions move on together. In this way the coupling time for the two diffusions is the same as the first time $Z^q_t=0$ (the first passage time T_0 to level 0 given by $T_0:=\inf\{t\geq 0, Z_t=0\}$) for either $Z^q_0=z$ or $Z^q_0=-z$ where $z=\varepsilon_q\cdot\eta$ (here $\varepsilon_1,\cdots,\varepsilon_q$

constitute the canonical orthonormal basis for $\mathbb{R}^q$ and " $\cdot$ " is the usual scalar product). We know from coupling theory that

$$d'(\eta,\eta') \leq 2\mathbb{P}^{\frac{x}{2}}(T_0 \geq t),$$

where $x = d(\eta,\eta')$ is the Euclidean distance between η and η'. Further it follows from the reflection principle of Désiré André ([15],pp.79-81 and [**?**],p.293)that

$$\mathbb{P}^{\frac{x}{2}}(T_0 \leq t) = 2\mathbb{P}^{\frac{x}{2}}(Z_t^q \leq 0) = 2\int_{-1}^{0} k_t\Big(\frac{x}{2}, s\Big)(1-s^2)^{\frac{q-3}{2}}ds.$$

The heuristic argument for the first equality in the above equation is as follows; the probability that the first passage time T_0 (to a level 0 for a 1-dimensional diffusion starting at some initial point $y > 0$) is less or equal to t is the sum of the probabilities of the events that $T_0 \leq t$ and $Z_t^q < 0$, and $T_0 \leq t$ and $Z_t^q > 0$. The probability for the first event is the same as the probability for the event that the 1-dimensional diffusion Z_t^q starting at y is below the level 0. For the probability of the second event observe that after the diffusion reached level 0, it has equal probability to reach level $-c$ below 0 or level c above 0 since the diffusion in our set-up is symmetric about 0. Hence the probability of the second event is the same as the first due to the symmetry of Z_t^q about 0.

It follows from the orthogonality property of the Legendre polynomials that for each positive even integer n the integral $\int_{-1}^{0} P_n(q,s)(1-s^2)^{\frac{q-3}{2}}ds = \frac{1}{2}\int_{-1}^{1} P_n(q,s)P_0(q,s)(1-s^2)^{\frac{q-3}{2}}ds = 0$ (since $P_0(q,s) = 1$) for all $q \geq 2$. Therefore the rest of the proof follows from (94) and the fact that the integral $\int_{-1}^{1} P_0(q,s)^2(1-s^2)^{\frac{q-3}{2}}ds = \frac{\sqrt{\pi}\Gamma\left(\frac{q-1}{2}\right)}{\Gamma\left(\frac{q}{2}\right)}$.

□

We have seen from the above proof that for positive even integers n the integral (over [-1,0] and w.r.t to the invariant measure $(1-s^2)^{\frac{q-3}{2}}ds$) of the Legendre polynomial of degree n is always equal to zero, as long as the dimension $q \geq 2$. The integral for the corresponding odd degree case can also be computed explicitly and this explicit value of the integral we formulate as our next lemma.

Lemma 5.5 *For any odd integer $2m+1$ (m=0,1,2,....) the integral of the Legendre polynomials $P_{2m+1}(q,\cdot)$ over the interval [-1,0] is given by*

$$\int_{-1}^{0} P_{2m+1}(q,s)(1-s^2)^{\frac{q-3}{2}}ds = (-1)^m \prod_{i=0}^{m}\left(\frac{2i-1}{q+2i-1}\right). \tag{97}$$

Proof: We obtain from definition of $P_{2m+1}(q,s)$ in Definition 5.3 that the integral

$$\begin{aligned}&\int_{-1}^{0} P_{2m+1}(q,s)(1-s^2)^{\frac{q-3}{2}}ds \\ &= \frac{-1}{2^{2m+1}\prod_{i=0}^{2m}\left(2m+\frac{q-1}{2}-i\right)}\Big(\frac{d}{ds}\Big)^{2m}\Big(1-s^2\Big)^{2m+\frac{q-1}{2}}\Big|_{s=-1}^{0}.\end{aligned} \tag{98}$$

Note that for each m the above differentiation(s) will always involve terms which are multiples of $(1-s^2)$. This implies that evaluating the above expression at $s=-1$ will always yield zero. However, it follows from Binomial expansion of $\left(1-s^2\right)^r$ (where $r=2m+\frac{q-1}{2}$) that

$$\begin{aligned}&\frac{-1}{2^{2m+1}\prod_{i=0}^{2m}\left(2m+\frac{q-1}{2}-i\right)}\left(\frac{d}{ds}\right)^{2m}\left(1-s^2\right)^{2m+\frac{q-1}{2}}\Big|_{s=0}\\&=\frac{(-1)^{m+1}(2m)!r(r-1)\cdots(r-(m-1))}{m!2^{2m+1}\prod_{i=0}^{2m}\left(2m+\frac{q-1}{2}-i\right)}.\end{aligned}\tag{99}$$

The rest of the proof follows from the observations that $(2m)!=2^m m!\prod_{i=1}^m(2i-1)$ and $\frac{r(r-1)\cdots(r-(m-1))}{\prod_{i=0}^{2m}\left(2m+\frac{q-1}{2}-i\right)}=\frac{2^{m+1}}{\prod_{i=0}^{m}(2i+q-1)}$.

□

Proof of the Proposition 2.8 :

1. It follows from Lemma 5.2 that, for any $q\geq 2$,

$$F_{q,t}(x)\leq 2\mathbb{P}^{\frac{x}{2}}\left(T_0\geq t\right)\leq 4P\Big(0\leq G\leq\frac{\arcsin\frac{x}{2}}{\sqrt{2t}}\Big).\tag{100}$$

   Using $P\Big(0\leq G\leq u\Big)\leq\frac{u}{\sqrt{2\pi}}$ by concavity and $\arcsin y\leq\frac{\pi}{2}y$ for $0\leq y\leq 1$ we obtain $F_{q,t}(x)\leq\frac{\sqrt{\pi}x}{2\sqrt{t}}$. Note that in both of the last estimates the constants were sharp.

2. The claim for general dimensions $q\geq 2$ follows from Lemma 5.4 and 5.5.

□

Proof of Theorem 2.9: This Theorem is an application of Theorem 2.6. The only quantities we have to worry about are the entries of the Dobrushin interdependence matrix $\bar{C}$. It follows from the hypothesis of the Theorem; namely the continuity property of the interaction and the terms in bound on $c'[\eta]$ in Corollary 4.4 that

$$\begin{aligned}&\frac{1}{2}\exp\Big(\frac{1}{2}\sum_{A\supset\{i,j\}}\delta(\Phi_A)\Big)L_{ij}\inf_{a_i\in S_{s'}}\Big(\int_{S_{s'}}d^2(\sigma_i,a_i)\alpha_{s'}(d\sigma_i)\Big)^{\frac{1}{2}}\\&\leq\sup_{s'\in S'}\frac{\rho_{s'}}{2}\exp\Big(\frac{1}{2}\sum_{A\supset\{i,j\}}\delta(\Phi_A)\Big)L_{ij}=\bar{C}_{ij},\end{aligned}\tag{101}$$

where $\rho_{s'}:=\mathrm{diam}(S_{s'})$ is the diameter of $S_{s'}$.

□

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